Problem 1 J. The numerator and denominator of Juan's fraction are positive integers whose sum is 2011. The value of this fraction is less than $\frac{1}{3}$. Find the greatest such fraction.

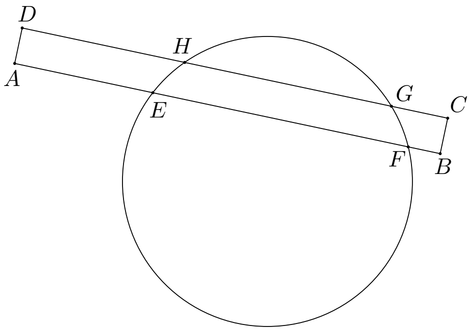
Answer. $\frac{502}{1509}$.

Solution outline. We clearly have $a < 2001$, so the relation can be rewritten as

$$\frac{a}{2011 - a} < \frac{1}{3}.$$

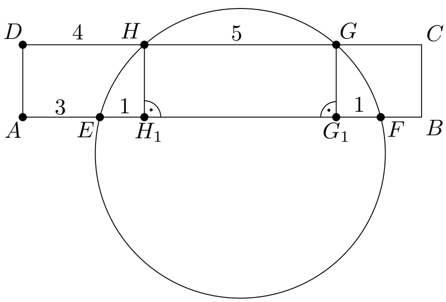
This rewrites as $4a < 2011$, from which we get that the biggest value of a is 502, so the smallest value of b is 1509, giving us the answer $\frac{502}{1509}$.

Problem 2 J. Rectangle $ABCD$ intersects a circle at points E, F, G, H as shown in the diagram. If $AE = 3$, $DH = 4$, and $GH = 5$, find EF.



Answer. 7.

Solution outline.



Let G_1, H_1 be the projections of G and H on the line AB. Then we obtain $H_1G_1 = HG = 5$. We also have $EH_1 = G_1F$ and $EH_1 = DH - AE = 1$, thus $EF = EH_1 + H_1G_1 + G_1F = 7$.

Problem 3 J. Find the sum of digits of the number $1+11+101+1001+10001+\cdots+1\underbrace{0\ldots0}_{50}1$.

Answer. 58.

Solution outline. Our number is equal to

$$1+11+101+1001+\cdots+1\underbrace{0\ldots0}_{50}1=$$
$$=1+(10+1)+(100+1)+(1000+1)+\cdots+(1\underbrace{0\ldots0}_{51}+1)=\underbrace{11\ldots11}_{51}0+52=\underbrace{11\ldots11}_{50}62.$$

From this we get that the sum of digits of our number is 58.

Problem 4 J. T-shirts were made in three colors, red, gray and blue. The number of red T-shirts is six smaller than the total number blue and gray T-shirts. Also, the number of gray T-shirts is ten smaller than the total number red and blue T-shirts. How many blue T-shirts were made?

Answer. 8.

Solution outline. Let a, b, c be the number of red, white and respectively blue T-shirts. We then obtain the system of equations

$$a=b+c-6,$$
$$b=a+c-10.$$

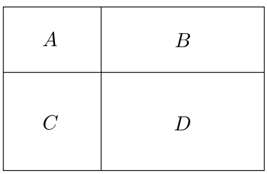
By adding the two relations we obtain $c=8$.

Problem 5 J. There are 33 walnuts on the table in at least two piles. Each pile consists of at least 2 walnuts. After moving one walnut from each pile to the first one, all piles will have the same number of walnuts. What was the original number of piles? Find all possibilities.

Answer. 3.

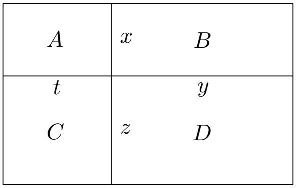
Solution outline. Let $p_1, p_2, \ldots p_{n+1}$ be all the piles. Then $p_2=p_3=\cdots=p_{n+1}=b$, and let $p_1=a$. We then obtain $a+nb=33$ and $a+n=b-1$, and by subtracting these two relations we get $bn-n=34-b$, so $(b-1)(n+1)=33$. If $n+1=11$, then $b=4$, and we cannot have $a+n=b-1$, so we are only left with $n+1=3$, which works.

Problem 6 J. A rectangle is divided by two segments parallel to its sides into four smaller rectangles. Let's label them A, B, C, D as in the diagram. Given that the perimeters of rectangles A, B, C are 2, 4, and 7, respectively, find all possible values of the perimeter of rectangle D.



Answer. 9 cm.

Solution outline.



Let x, y, z, t be the lengths of the sides of the four rectangles, as in the figure, and let o_A, o_B, o_C, o_D be the perimeters of the rectangles A, B, C, D respectively. Then

$$o_A + o_D = 2(t + x) + 2(y + z) = 2(x + y) + 2(t + z) = o_B + o_C,$$

therefore $o_D = 4 + 7 - 2 = 9$ cm.

Problem 7 J. Find pairwise distinct digits A, B, C such that

$$\begin{array}{r} A \\ AB \\ \underline{ABC} \\ BCB \end{array}$$

Answer. $A = 6$, $B = 7$, $C = 4$.

Solution outline. Let's write, for conveniance, $a = A$, $b = B$, $c = C$. Then we get $a + 10b + c + 100a + 10b + c = 100b + 10c + b$, so $111a = 90b + 9c$. This means that 9 divides $111a$, so 3 divides a.

If $a = 3$, then $b = 3$ and $c = 7$, so a, b, c are not pairwise distinct.

If $a = 6$, then $b = 7$ and $c = 4$.

If $a = 9$, then we obtain $111 = 10b + c$, which doesn't satisfy the property that b and c are digits. So the only solution is $a = 6$, $b = 7$, $c = 4$.

Problem 8 J. Find the area of a rectangle with perimeter 10 cm and diagonal $\sqrt{15}$ cm.

Answer. 5 cm^2.

Solution outline. Let a and b be the sides of our rectangle. Then we obtain $2(a+b) = 10$, $a^2+b^2 = 15$, so

$$2ab = (a + b)^2 - \left(a^2 + b^2\right) = 5^2 - 15 = 10.$$

Therefore the area is 5 cm^2.

Problem 9 J. Andrei took N^3 equally sized white cubes and used them to form one big cube $N \times N \times N$. Then he colored its surface in red. Given that now one tenth of the total surface of the cubes is red, find N.

Answer. 10.

Solution outline. Let S be the area of one face of a small cube. Then the sum of the surface areas of all the cubes is $6N^3S$, and the surface area of the big cube is $6N^2S$. The problem statement tells us that $6S \cdot N^2 = \frac{6S}{10}N^3$, so $N = 10$.

Problem 10 J. What is the least possible number of members of a math circle, where girls form more than 48.5% but less than 50% of the members?

Answer. 35.

Solution outline. Let g be the number of girls in the math circle, and let N be the total number of people in the circle. Then the problem statement tells us that $\frac{48{,}5}{100}N < g < \frac{N}{2}$, so we can write

$N = 2g + a$, with a an integer, $a > 0$, and $97N = 200g - b$, for an integer b with $b > 0$. The last equation can be rewritten as $97(2g + a) = 200g - b$, or equivalently $97a + b = 6g$. If $a = 1$, then the smallest value of b which satisfies our equation is $b = 5$, which gives us $g = 17$ and $N = 35$.

If $a \geq 2$, then $6g > 194$, so $g \geq 33$, which means $N \geq 67$. Therefore $a = 1$ gives us the least value of N, which is indeed 35.

Problem 11 J / 1 S. If you increase the number of this problem by n, you obtain the number of the most shocking problem. Whereas if you increase it by a two-digit number k, you obtain the number of the most playful problem. Moreover, we have $n^3 = k^2$. Find n and k, given that you have 30 problems left (including this one).

Answer. $n = 9$, $k = 27$.

Solution outline. For k^2 to be a cube, k must be a cube itself. The only two-digit cube less than 41 is 27, which gives us $n = 9$.

Problem 12 J / 2 S. Find positive integer n such that $6666^2 + 8888^2 = n^2$.

Answer. 11110.

Solution outline. We regroup the terms to get

$$n = \sqrt{1111^2 \cdot 6^2 + 1111^2 \cdot 8^2} = 1111\sqrt{36 + 64} = 11110.$$

Problem 13 J / 3 S. Find the smallest positive integer, which ends with number 17, which is divisible by 17, and whose sum of digits is 17.

Answer. 15317.

Solution outline. We write the number in the form $100 \cdot a + 17$ for some $a \in \mathbb{N}_0$. Since 17 is coprime with 100, we get that a is divisible by 17 and has sum of digits 9. Thus it is also a multiple of 9. Smallest such a is 153, thus the solution is 15317.

Problem 14 J / 4 S. Each pair of consecutive digits of a 2011-digit number is a multiple of either 17 or 23. Its last digit is 1. Find its first digit.

Answer. 3.

Solution outline. Write all two-digit multiples of 17 and 23 to find out that each unit digit appears exactly once. In this way we can restore the original number from its end as $\ldots 92346\,92346\,851$. Noticing the periodicity, we easily calculate that the first digit is three.

Problem 15 J / 5 S. A positive integer is called *awesome* if any other positive integer with the same sum of digits is greater. How many three-digit awesome numbers exist?

Answer. 9.

Solution outline. Realize that for each value $k \in \mathbb{N}$ of the sum of the digits there is exactly one smallest positive integer with this sum of its digits, i.e. one awesome number. Let's denote it a_k. The number $a_1 \ldots a_{18}$ will be two digit integers, whereas $a_{28}, a_{29}, \ldots$ will have more than three digits. Also the sums of digits of three-digit integers attain all numbers between 19 and 27, thus $a_{19}, \ldots, a_{27}$ will indeed have three digits and thus we have 9 awesome three-digit numbers.

Problem 16 J / 6 S. Tim has found real numbers x, y, z satisfying $\frac{x-y}{z-y} = -10$. What are the possible values of $\frac{x-z}{y-z}$?

Answer. 11.

Solution outline.

$$\frac{x-z}{y-z} = \frac{(x-y)+(y-z)}{y-z} = \frac{x-y}{y-z} + \frac{y-z}{y-z} = 10 + 1 = 11.$$

Problem 17 J / 7 S. The numbers $1, 2, \dots, 9$ are arranged in some order to form a nine-digit integer. Consider all triplets of consecutive digits and add the corresponding seven three-digit numbers. What is the largest result that can be obtained?

Answer. 4648.

Solution outline. Let the digits be a_1, a_2, ... , a_9. The triplet then equal:

$$100a_1 + 10a_2 + a_3, \quad 100a_2 + 10a_3 + a_4, \quad 100a_3 + 10a_4 + a_5, \quad \dots, \quad 100a_7 + 10a_8 + a_9,$$

so their sum equals

$$100a_1 + 110a_2 + 111a_3 + \dots + 111a_7 + 11a_8 + 1a_9.$$

To maximize the sum we set a_3 to a_7 the highest digits and $a_2 = 4$, $a_1 = 3$, $a_8 = 2$, $a_9 = 1$. The result is

$$111 \cdot (5 + \dots + 9) + 4 \cdot 110 + 3 \cdot 100 + 2 \cdot 11 + 1 \cdot 1 = 4648.$$

Problem 18 J / 8 S. A real number is written in each cell of a 10×10 square. Emily wrote down all products of two numbers from two distinct cells of the table and noticed that exactly 1000 of these products were negative. How many times did number 0 appear in the original square? Find all possibilities.

Answer. 30, 35.

Solution outline. Let p, n be the numbers of positive and negative numbers written in the square, respectively. We observe that we must have $p + n \leq 100$ and $p \cdot n = 1000$, which has two solutions $20, 50$ and $25, 40$, which lead to the number of zeros equal to 30 or 35.

Problem 19 J / 9 S. Math kingdom started to produce a new set of coins. On the first day they created coins with value 1 MD (Math Dollar). Every other day they created coins with the smallest value which cannot be paid by at most ten existing coins. Which coins did they create on the 2011th day?

Answer. 20101.

Solution outline. Prove by induction that during the k-th day they created coins with value $10(k-1) + 1$. The result follow by plugging in $k = 2011$.

Problem 20 J / 10 S. Let the number p be the solution to this problem. Find the probability that a randomly chosen point inside a unit square is at least p units away from all sides.

Answer. $\frac{1}{4}$.

Solution outline. A point is not closer than p to any of its sides if and only if it lies inside a small square with sidelength $1 - 2p$. We calculate the probability as the ratio of areas

$$\frac{(1-2p)^2}{1^2} = p.$$

This equation has two solutions: $p = \frac{1}{4}$ works, $p = 1$ does not.

Problem 21 J / 11 S. A 3×3 square is filled with integers such that the sums of the horizontal rows increase by two going downwards and the sums of the vertical columns double from left to right. Given that the sum of the numbers in one of the rows is 2011, find the sum of the numbers in the leftmost column.

Answer. 861.

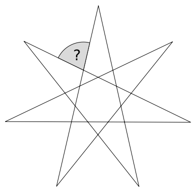
Solution outline. Let a be the sum of the numbers in the first row and let b be the sum of the numbers in the leftmost column. We calculate the sum of the numbers in the square in two ways to get $3a + 6 = 7b$. Distuingishing the three cases based on which row has the sum of the numbers 2011, we get the only integer solution $b = 861$.

Problem 22 J / 12 S. There are 2 boats, one on each side of a river bank. They both sail towards each other at a constant speed (not necessarily the same). The first time they meet, they are 100 meters from one side of the bank. Once they reach the side of the bank, they turn around and move towards each other again. This time they meet 70 meters from the other side of the bank. How wide is the river?

Answer. 230 m.

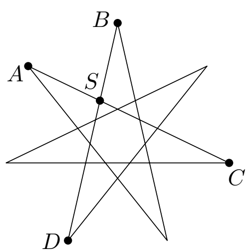
Solution outline. Let S be the length of the river bank. When the boats first met, they had travelled S meters altogether. When they met the second time, they had travelled $3S$ meters. For the boat which had travelled 100 meters before the first meeting, we can use the constant speeds to form the equation $3 \cdot 100 = S + 70$, which yields $S = 230$.

Problem 23 J / 13 S. Vertices of a star form a regular heptagon. What is the magnitude of the marked angle?



Answer. $\frac{3\pi}{7} = \frac{540°}{7} = 77° + \frac{1°}{7}$.

Solution outline.



Denote points A, B, C, D, S as in the diagram. If we rotate the segment AC counterclockwise about the center by angle $2\pi \cdot \frac{2}{7}$, we obtain the segment DB. Thus $\angle DSA = \frac{4\pi}{7}$ and ASB is supplementary.

Problem 24 J / 14 S. Find x such that $2^{2^{3^{2^{2}}}} = 4^{4^{x}}$.
Note: the order of operations is: $4^{3^2} = 4^9$.

Answer. 40.

Solution outline. Rewrite the righthand side:

$$4^{4^x} = \left(2^2\right)^{\left(2^2\right)^x} = 2^{2 \cdot 2^{2x}} = 2^{2^{2x+1}}.$$

So we need $2x + 1 = 3^{2^2} = 3^4 = 81$, or in other words $x = 40$.

Problem 25 J / 15 S. How many triplets of positive integers (a, b, c) exist, such that

$$\frac{\frac{a}{c} + \frac{a}{b} + 1}{\frac{b}{a} + \frac{b}{c} + 1} = 11$$

and $a + b + c \leq 30$?

Answer. 24.

Solution outline. Write $1 = \frac{a}{a}$ and $1 = \frac{b}{b}$ to get

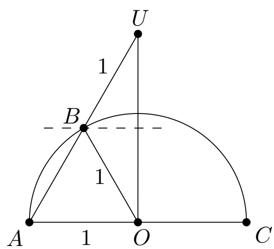
$$\frac{\frac{a}{c} + \frac{a}{b} + 1}{\frac{b}{a} + \frac{b}{c} + 1} = \frac{a\left(\frac{1}{c} + \frac{1}{b} + \frac{1}{a}\right)}{b\left(\frac{1}{a} + \frac{1}{c} + \frac{1}{b}\right)} = \frac{a}{b} = 11.$$

For $b = 1$, $a = 11$ we get $c = 1, 2, \ldots, 18$ and for $b = 2$, $a = 22$ we get $c = 1, 2, \ldots, 6$, which is 24 triplets altogether.

Problem 26 J / 16 S. A circle ω with radius 1, center O and diameter AC is given in the plane. Draw a line ℓ through point O such that it is perpendicular to AC. Choose a point U on ℓ such that U is outside ω. Denote by B the second intersection of AU and ω and assume $BU = 1$. Find OU.

Answer. $\sqrt{3}$.

Solution outline.



Since $OB = 1 = BU$, the triangle OUB is isosceles. Thus B lies on the perpendicular bisector of AOU and also on its hypotenuse, so it is the midpoint of the hypotenuse. Thus $AB = 1$ and Pythagorean theorem gives $OU = \sqrt{3}$.

Problem 27 J / 17 S. Two nations A and B are in a battle with 1000 soldiers involved altogether. The armies take turns to attack. In each turn every living soldier from the attacking army shoots a soldier from the enemy's army. The battle ended (not necessarily by elimination of one of the sides) after three turns (A was shooting first, then B and finally A again). What is the least guaranteed number of survivors?

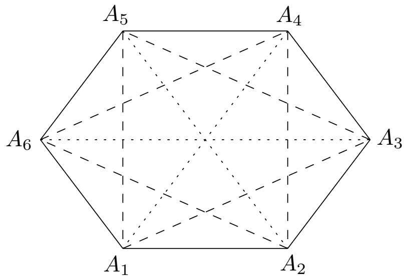
Answer. 200.

Solution outline. Assume n soldiers survived and deduce that there must have been at most $5n$ soldiers at the beginning of the battle. Also note that 200 survivors are indeed possible.

Problem 28 J / 18 S. All six sides of a convex hexagon $A_1A_2A_3A_4A_5A_6$ are colored in red. Each of the diagonals is either blue or red. Find the number of such colorings that each triangle $A_iA_jA_k$ $(i \neq j \neq k \neq i)$ has at least one red side.

Answer. $392 = 7 \cdot 7 \cdot 8$.

Solution outline. Apart from $A_1A_3A_5$ and $A_2A_4A_6$ (dashed in the diagram) all other triangles have a red edge. Every dashed triangle can be colored in $2^3 - 1 = 7$ ways. Finally, we have $2^3 = 8$ ways to color the dotted diagonals A_1A_4, A_2A_5, A_3A_6. Altogether, this gives us $7 \cdot 7 \cdot 8 = 392$ suitable colorings.



Problem 29 J / 19 S. Malcom told each Michal and Shri a positive integer. Further, he told them, that the numbers they heard were distinct and that their sum is a two-digit number. Then the following conversation took place:

Michal: "I cannot determine which one of us has the greater number."

Shri: "I can't determine it either, but I will tell you that my number is divisible by 17."

Michal: "Wow! Now, I can determine the sum of our numbers."

Find the value of this sum, given that the logic Michal and Shri used was flawless.

Answer. 51.

Solution outline. Since neither of them can determine whose number is greater, neither has a number greater than fifty. Moreover, since Shri's number is divisible by 17 it can be either 17 or 34. For Michal to be able to determine the sum of the two numbers, he must have the other one (the numbers are distinct!). Hence the sum is 51.

Problem 30 J / 20 S. There are 55 guests in a café, Turks and Indians. Each of them drinks either tea or coffee. An Indian speaks truth if he drinks tea and lies if he drinks coffee, whereas with the Turks it is the other way round. For questions "Do you drink coffee?", "Are you Turkish?" and "Is it raining outside?" the numbers of positive answers were 44, 33, and 22, respectively. How many Indians drink tea? Find all possibilities.

Answer. 0.

Solution outline. Form equations and find two solutions (based on whether it is raining outside or not) one of which does not work since there would be $\frac{11}{2}$ Indians drinking tea, which is against humanity. The other solution works and yields 0 Indians drinking tea.

Problem 31 J / 21 S. Three digits were written to the end of a positive integer A. The resulting number was equal to the sum of numbers from 1 to A. Find all possible values of A.

Answer. 1999.

Solution outline. Let B be the three digit-integer written to the end of A. Since $1+2+\cdots+A = \frac{1}{2}A(A+1)$, we have

$$1000A + B = \frac{A(A+1)}{2},$$

which we transform to $2B = A(A-1999)$. The left handside is between 0 and $2 \cdot 999$, whereas the righthand side is negative for $A \leq 1998$ and greater than $2000 \cdot 1$ for $A \geq 2000$. We are left with $A = 1999$ which satisfies the problem for $B = 000$.

Problem 32 J / 22 S. Alice, Betty, Claudia, Daniel, and Eli were playing doubles tournament in table-tennis. Each pair played against each other pair exactly once. Alice won 12 games and Betty won 6 games. How many games could Claudia win? Find all possibilities.

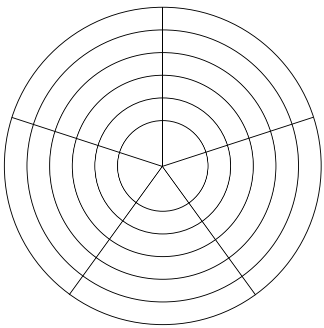
Answer. 4.

Solution outline. We calculate that each player plays in 12 games. This implies Alice won all of her games and also that Betty won all games which were not against Alice. This ensures we know the result of every game and we find that Claudia won 4 games.

Problem 33 J / 23 S. Two players are playing a game on the given plan consisting of 30 cells. The rules are the following.

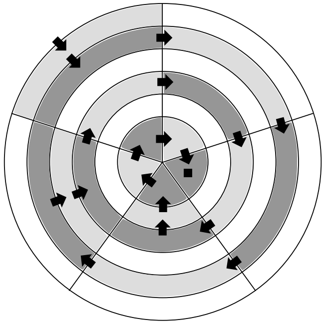
- players take turns,
- in one move a player colors one cell,
- in the first move, only a cell neighbouring with the edge can be colored. In any other moves, only a cell which is next to the last colored cell and is not further away from the center, can be colored,
- once a cell is colored, it cannot be colored again.
- the player who can no longer make a move, loses.

How many cells will be colored by the end of the game, in which both players play perfectly and the one who cannot win tries to make the game as long as possible?



Answer. 18.

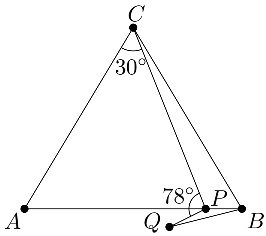
Solution outline. Show that the second player has a winning strategy. The game is going to look something like this with 18 moves.



Problem 34 J / 24 S. In triangle ABC with $AC = BC$, we find a point P ($P \neq B$) on the side AB, such that $PB < PA$ and $\angle ACP = 30°$. Further, we find point Q such that $\angle CPQ = 78°$ and points C and Q lie each on opposite side of AB. If the internal angles in triangles ABC and BQP have integral values (in degrees), find all possible values of angle BQP.

Answer. $1°$.

Solution outline.



Since $|\sphericalangle PAC| = |\sphericalangle CBA| < |\sphericalangle CPA|$ and $|\sphericalangle PAC| + |\sphericalangle CPA| = 180° - 30° = 150°$, we must have $|\sphericalangle PAC| < 75°$. Using integral value of the angle we even get that $|\sphericalangle PAC| \leq 74°$ a $|\sphericalangle CPA| \geq 76°$.

But also $|\sphericalangle APQ| = |\sphericalangle PBQ| + |\sphericalangle BQP| \geq 1° + 1° = 2°$, thus

$$78° = |\sphericalangle CPQ| = |\sphericalangle CPA| + |\sphericalangle APQ| \geq 76° + 2° = 78°.$$

So we must have equality everywhere, which yields $|\sphericalangle BQP| = 1°$.

Problem 35 J / 25 S. Ten people in the theater are sitting next to each other in one row. After a break they sit in a new arrangement, so that only two people remained in their original positions and the remaining eight sat next to their former position. In how many ways could they have done that?

Answer. $15 = \binom{6}{2}$.

Solution outline. Show that any such seating can be represented by a sequence of six letters two of which are S representing people who stay in their original position and four of which are C representing a pair which changes (switches) their seats. There are $\binom{6}{2} = 15$ such sequences.

Problem 36 J / 26 S. A positive integer is written on each face of a cube. To each vertex we assign the product of the numbers written on the three faces intersecting at that vertex. The sum of the numbers assigned to the vertices is 165. What are the possible values of the sum of the numbers written on the faces?

Answer. 19.

Solution outline. Denote the number on the faces by a, b, c, d, e, f so that the pairs a and f, b and e, c and d are opposite on the cube. Then

$$3 \cdot 5 \cdot 11 = 165 = (a+f)(b+e)(c+d),$$

where the second equality is veryfied by expanding. Using that a to f are positive integers, we get that the sum equals $3+5+11=19$. Such numbering can easily be found.

Problem 37 J / 27 S. Two bicyclists are racing on a straight path at constant speeds. They both start at one end and everytime they hit the end of the path, they turn around and go the opposite direction. Eventually, they meet again at one of the endpoints. Before that, the slower bicyclist had traveled the path 35 times (in one of the directions) and the faster one 47 times. How many times did they meet head on?

Answer. 40.

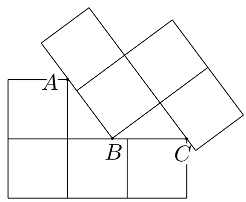
Solution outline. A head on meeting requires two changes of direction to be made (regardless if one bicyclist changes direction twice r both once). There are $34+46=80$ changes of direction, which means 40 head on meetings.

Problem 38 J / 28 S. Find the largest positive integer such that all its digits (other than the first one and the last one) are smaller than the arithmetic mean of the two surrounding digits.

Answer. 96433469.

Solution outline. Show that there cannot be more than 4 consecutively decreasing or increasing digits. Next show that the desired number consists of a decreasingsection of digits followed by an increasing section. Playing aroung with such 8-digit numbers gives the result 96433469.

Problem 39 J / 29 S. Two tetrominoes made of 1×1 squares touch at points A, B, C as in the diagram. Find the distance AB.



Answer. $\frac{5}{4} = 1{,}25$.

Solution outline. Right triangle with hypotenuses AB and BC are congruent by ASA. Let x be their shorter leg. Then $2 = x + |BC| = x + |AB|$ a by Pythagorean theorem also $x^2 + 1 = |AB|^2$. Solving the equations gives $|AB| = 1,25\,\mathrm{dm}$.

Problem 40 J / 30 S. There are 100 points with integral coordinates given in the plane. We connect each pair with a segment. How many of these segments are guaranteed to have a midpoint with integral coordinates?

Answer. $1200 = 4 \cdot \binom{25}{2}$.

Solution outline. The midpoint has integer coordinates if and only if the x-coordinates of the two points have the same parity as well as the y-coordinates. Now divide the points in four groups depending on the parity of its x and y coordinates. Use power mean inequality to show that extreme case is when all groups have same size. Each group gives you $\binom{25}{2}$ midpoints with integers coordinates so the answer is $4 \cdot \binom{25}{2} = 1200$

Problem 41 J / 31 S. A five-digit integer is called *irreducible* if it cannot be written as a product of two three-digit integers. What is the maximum possible number of consecutive irreducible integers?

Answer. 99.

Solution outline. To be translated.

Problem 42 J / 32 S. Real numbers x and y satisfy $(x+5)^2 + (y-12)^2 = 14^2$. Find the minimum possible value of $x^2 + y^2$.

Answer. 1.

Solution outline. To be translated.

Problem 43 J / 33 S. A sequence is definned as follows: $a_1 = 20$, $a_2 = 11$, and

$$a_{n+2} = a_n - \frac{1}{a_{n+1}},$$

as long as the right-hand is well-defined. Find the least t such that $a_t = 0$.

Answer. 222.

Solution outline. To be translated.

Problem 44 J / 34 S. Let ABC be an acute-angled triangle with altitudes AA', BB', CC', intersecting at H. Given that

$$\frac{|AH|}{|HA'|} = 1, \quad \frac{|BH|}{|HB'|} = 2,$$

find $\frac{|CH|}{|HC'|}$.

Answer. 5.

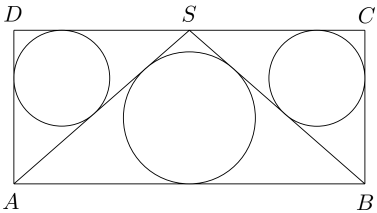
Solution outline. To be translated.

Problem 45 J / 35 S. Every guest at a party (including Tim) knows exactly seven boys and ten girls there. What is the least possible number of people at the party?

Answer. 34.

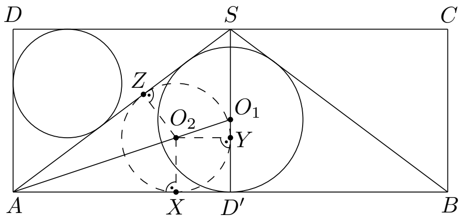
Solution outline. To be translated.

Problem 46 J / 36 S. Let S be the midpoint of side CD of rectangle $ABCD$. The incircles of triangles ASD and BSC have radii both equal to 3 and the inradius of $\triangle ASB$ is 4. Find the sides of the rectangle.



Answer. 9, 24.

Solution outline.



To be translated.

Problem 47 J / 37 S. We write out all divisors of a positive integer n which are less than n from the greatest to the lowest. If n is the sum of the second and third divisor, we say that n is *additive*. How many additive numbers are there which are less than 15000?

Answer. 1000.

Solution outline. To be translated.

Problem 48 J / 38 S. Find all real numbers x such that

$$\frac{x-49}{50}+\frac{x-50}{49}=\frac{50}{x-49}+\frac{49}{x-50}.$$

Answer. 99, 0, $\frac{4901}{99}=49\frac{50}{99}$.

Solution outline. To be translated.

Problem 49 J / 39 S. A position of a minute hand and an hour hand on the clock is called *valid*, if it might occur during one 12 hour cycle. Find the number of valid positions which remain valid after switching the two hands.

Answer. 143.

Solution outline. To be translated.

Problem 50 J / 40 S. Let a, b, c be nonzero real numbers, such that the quadratic equations $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root. Find all possible real values of this root.

Answer. 1.

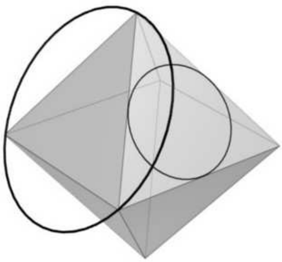
Solution outline. To be translated.

Problem 51 J / 41 S. Find all integers n such that both $16n+9$ and $9n+16$ are perfect squares.

Answer. 0, 1, 52.

Solution outline. To be translated.

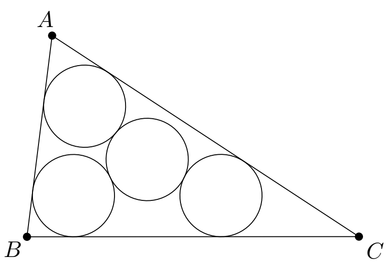
Problem 52 J / 42 S. A regular octahedron with side length 2 is given in space. One circle is inscribed in one of the faces and another circle is circumscribed about an adjacent face. What is the minimal distance between the two circles?



Answer. $\sqrt{2} - 1 = \sqrt{3 - 2\sqrt{2}}$.

Solution outline. To be translated.

Problem 53 J / 43 S. Let ABC be a triangle with circumradius 5 and inradius 2. Three circles with radius r are inscribed in angles BAC, CBA, ACB, respectively such that the circles are in the interior of the triangle and there exists another circle with radius r which is tangent to all three circles. Find r.



Answer. $\frac{10}{9}$.

Solution outline. To be translated.

Problem 54 J / 44 S. Real numbers a, b, x, y satisfy

$$\begin{aligned} ax + by &= 3, \\ ax^2 + by^2 &= 7, \\ ax^3 + by^3 &= 16, \\ ax^4 + by^4 &= 42. \end{aligned}$$

Find $ax^5 + by^5$.

Answer. 20.

Solution outline. To be translated.