Problem 1J. Peter has a gold bar of size $2 \times 3 \times 4$. Being an amateur blacksmith, he melted down the bar and created three identical cubes from the liquid gold. What is the common sidelength of Peter's cubes?

Result. 2

Solution outline. The volume of the original blockstone was $2 \cdot 3 \cdot 4 = 24$. Since all three cubes have the same volume, it equals $\frac{24}{3} = 8$. As the volume of a cube is the third power of its sidelength, we conclude that the sidelength of the three golden cubes is $\sqrt[3]{8} = 2$.

Problem 2J. A new Year's Eve celebration was attended by 43 people. The bar was selling juice, beer, and champagne. During the night 25 people drank beer, 19 people drank champagne, and 12 people drank both beer and champagne. The others were drivers, so they drank only juice. How many people drank only juice?

Result. 11

Solution outline. The total number of people drinking alcohol was $25 + 19 - 12 = 32$ (we added the numbers of people drinking beer and people drinking champagne and subtracted the number of those drinking both beverages, since we counted them twice). All the others drank juice only, so there were $43 - 32 = 11$ such people.

Problem 3J. By drinking a cup of black tea, one gets enough caffeine for one hour. By drinking a cup of coffee, one gets caffeine for 4 hours. In what ratio should we mix black tea and coffee in order to have a full cup containing caffeine for two hours?

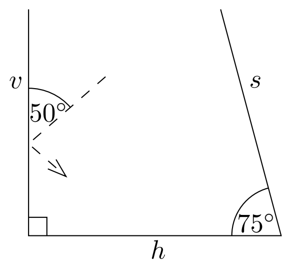
Result. $2 : 1$

Solution outline. Denote by k the amount of caffeine for one hour. Then black tea contains k, coffee contains $4k$ and our goal is to mix $2k$ into one cup. If we fill a fraction x of a cup with tea and the remainder $1 - x$ with coffee into one cup, the amount of caffeine in the mixture satisfies $x \cdot k + (1 - x) \cdot 4k = 2k$. This yields $x = 2/3$ and hence the desired ratio is

$$\frac{x}{1-x} = \frac{\frac{2}{3}}{1 - \frac{2}{3}} = 2 : 1.$$

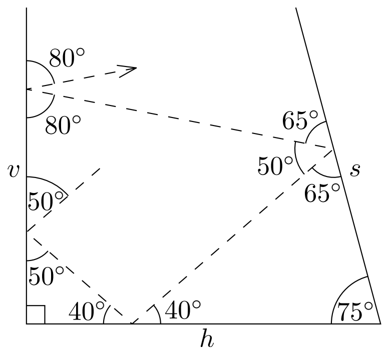
Problem 4J. The mirrors v and h in the picture are perpendicular to each other. The angle between the mirrors h and s is $75°$. A light ray was incident on the mirror v, from which it reflected to h, then to s, and finally back to v again. The initial angle of incidence on v was $50°$. What was the angle of incidence on v after reflecting from s?

Note: The angle of reflection equals the angle of incidence.



Result. $80°$

Solution outline. We sketch the trajectory of the light ray into a picture and compute angles in the resulting triangles. The angle of the first reflection from v is $50°$ from the last assumption. We know that the sum of angles in a triangle is $180°$ and v is perpendicular to h. Therefore the angle of incidence on h was $40°$, as was the angle of reflection. By the same reasoning, we infer that the angle of reflection from s was $65°$. Next we recall that the sum of angles in a quadrilateral is $360°$ and hence compute the desired value, i.e. $360° - 90° - 75° - (180° - 65°) = 80°$.



Problem 5J. A shop sold 235 intergalactic spaceships during the year 2013. In each month, 20, 16, or 25 spaceships were sold. Determine in how many months the shop sold exactly 20, 16, and 25 spaceships, respectively.

Result. 4, 5, 3

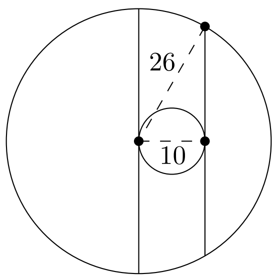
Solution outline. Let a, b and c be the number of months in which 20, 16 and 25 spacecrafts were sold, respectively. Then $20a + 16b + 25c = 235$. The right hand side is divisible by 5, so the left hand side has to be divisible by 5, too. Clearly, $20a + 25c$ is a multiple of 5, hence $16b$ has to be a multiple of 5 as well. Since $5 \nmid 16$, necessarily $5 \mid b$. There are only three possibilities:

- $b = 0$. Then $c = 12 - a$ and we get $20a + 25(12 - a) = 235$ or $4a + 60 - 5a = 47$. Thus $a = 13$, which is not possible as there are only twelve months altogether.
- $b = 5$. Then $a = 7 - c$ and substitution gives $80 + 5c = 95$. The solution is $a = 4$, $c = 3$.
- $b = 10$. Then $c = 2 - a$ and $210 - 5a = 235$, which is not possible.

Problem 6J. Two circles have radii 5 and 26. The centre of the larger circle lies on the circumference of the smaller one. Consider the longest and the shortest chord of the larger circle which are tangent to the smaller circle. What is the difference between the lengths of these extremal chords?

Result. 4

Solution outline. The longest chord is clearly the diameter of the larger circle, so its length is 52. The shortest is the one whose distance from the centre of the larger circle is the greatest. This distance can be at most 10, because the chord has to be tangent to the smaller circle. By the Pythagorean theorem, the length of the shortest chord is $2 \cdot \sqrt{26^2 - 10^2} = 48$, so the desired difference is $52 - 48 = 4$.

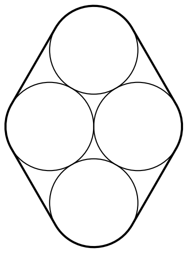


Problem 7J. Collin Farrel was born in the last century in Sleepy Hollow. We know that his age in 1999 was the same as the sum of the digits of the year when he was born. What is Collin's year of birth?

Result. 1976

Solution outline. Let $10x + y$ be his age in 1999, where x, y are digits. Since he was born in the 20th century, the sum of the digits of his year of birth is $1+9+(9-x)+(9-y)$, therefore $28-x-y = 10x+y$ or $28 = 11x+2y$. Since $0 \le x, y \le 9$, the only solution is $x = 2$, $y = 3$, hence he was born in $1999-23 = 1976$.

Problem 8J. Four circles of radius 1 are pairwise tangent (except for the top and the bottom one) as in the figure below. A tight band is wrapped around them, as shown in the figure. What is the length of the band?



Result. $8 + 2\pi$

Solution outline. We can split the band into straight and curved parts. The curved parts together form a full turn, so their total length is 2π. The length of each straight part is the same as the distance of the centers of the corresponding circles, which is 2. Thus the overall length is $8 + 2\pi$.

Problem 9J. A certain sports teacher always orders his class of bugs by the number of their legs. In the last sports session, only Buggy and four other bugs, who have 6, 3, 10, and 9 legs, were present. We know that when they stood in the correct order, the number of legs of the middle bug was the arithmetic mean of the numbers of legs of all five bugs. Find all possible numbers of Buggy's legs.

Result. 2, 7, 17

Solution outline. Let m be the number of legs of the middle bug, and let b be number of Buggy's legs. Then we have

$$m = \frac{6 + 3 + 10 + 9 + b}{5} = 5 + \frac{3 + b}{5}$$

or $b = 5m - 28$. As m is also the median (i.e. the "middle value"), it has to be one of 6, 9, or b, so (by plugging into the last equation) we find that b is one of 2, 7, or 17.

Problem 10J. For non-zero digits X and Y, denote $\overline{XY}$ the number whose decimal representation is XY. Assuming non-zero digits A, B, C satisfy $\overline{AA} + \overline{BB} + \overline{CC} = \overline{ABC}$, determine $\overline{ABC}$.

Result. 198

Solution outline. Comparison of the units digit in the equality yields $A + B + C = C + f$ or $A + B = f$, where f is 0 or 10; as A, B are non-zero digits, only $A + B = 10$ is possible. Similarly, the tens digit gives $A + B + C + 1 = B + 10$ (the one on the left hand side is carried from units) or $A + C = 9$. Finally, since we again carry only 1 to the hundreds, we obtain $A = 1$, thus $B = 9$ and $C = 8$.

Problem 11J / 1S. The famous celestial object collector Buggo was forced to sell one third of his collection due to the financial crisis. Immediately afterwards he gave three of his Solar System planets to his daughter. Later, he sold one third of the rest of his collection, and, furthermore, he also gave two moons of Jupiter and two moons of Saturn to his wife. Finally he sold one third of the rest of his objects followed by Mars with both its moons, and after that there were only 9 planets in the Alpha Centauri system left in his collection. How many objects did Buggo have at the beginning?

Result. 54

Solution outline. Let x be the number of celestial objects at the beginning. The number of objects changes during the process in the following way:

$$x,\ \tfrac{2}{3}x,\ \tfrac{2}{3}x - 3,\ \ldots,\ \tfrac{2}{3}\big(\tfrac{2}{3}\big(\tfrac{2}{3}x - 3\big) - 4\big) - 3 = 9.$$

After a bit of manipulation we obtain

$$x = \tfrac{3}{2}\big(\tfrac{3}{2}\big(\tfrac{3}{2}(9 + 3) + 4\big) + 3\big) = 54.$$

Problem 12J / 2S. Find the least positive integer greater than 2014 that cannot be written as a sum of two palindromes.

Note: A palindrome is a number whose digits (in decimal representation) are the same read backward.

Result. 2019

Solution outline. If a number greater than 2014 can be written as a sum of two palindromes, then one of these palindromes needs to be at least 4 digits long. However, there are not many of these – namely $1001, 1111, \ldots, 1991, 2002$ (and the rest is too large). We can write the numbers up to 2018 as $2015 = 1551 + 464$, $2016 = 1441 + 575$, $2017 = 1331 + 686$ and $2018 = 1221 + 797$.

Assume that 2019 can be decomposed in such a fashion; then one of the summands must have four digits. If the second summand had an even number of digits, their sum would be a multiple of 11, whereas 2019 is not. Thus we have $2019 = \overline{1AA1} + \overline{BCB}$, where A, B and C are digits. From the units digit we see that $B = 8$, so there is no carrying and either $A + C = 1$ or $A + C = 11$. The former would imply $\overline{1AA1} + \overline{8C8} < 2000$, the latter $A + 8 + 1 = 10$ (hundreds digit), so $A = 1$ and $C = 10$ which is a contradiction. Therefore the number we are looking for is 2019.

Problem 13J / 3S. Vodka gave Ondro a number puzzle. He chose a digit X and said: "I am thinking of a three digit number that is divisible by 11. The hundreds digit is X and the tens digit is 3. Find the units digit." Ondro soon realised that he had been cheated and there is no such number. Which digit X did Vodka choose?

Result. 4

Solution outline. Taking into account the well-known rule for divisibility by 11 the problem simplifies to determining for which digit X there is no digit Y (the units digit) such that $(X + Y) - 3$ is divisible by 11. This further reduces to seeking X such that $0 < X + Y - 3 < 11$ for all Y, which is satisfied only for $X = 4$.

Problem 14J / 4S. The real numbers a, b, c, d satisfy $a < b < c < d$. If we pick all six possible pairs of these numbers and compute their sums, we obtain six distinct numbers, the smallest four being 1, 2, 3, and 4. Determine all possible values of d.

Result. 3.5, 4

Solution outline. The two smallest sums clearly are $a + b$ and $a + c$, hence $a + b = 1$ and $a + c = 2$. Similarly, the two largest sums are $c + d$ and $b + d$, so there are two cases:

- $a + d = 3$, $b + c = 4$: Then we have $2b = (a + b) - (a + c) + (b + c) = 1 - 2 + 4$, so $b = 1.5$, resulting in $a = -0.5$ and $d = 3.5$.
- $b + c = 3$, $a + d = 4$: The same computation yields $b = 1$, thus $a = 0$ and $d = 4$.

Problem 15J / 5S. Jack is twice as old as Jill was when Jack was as old as Jill is today. When Jill is as old as Jack is now, the sum of their ages will be 90. How old is Jack now?

Result. 40

Solution outline. Let x and y be Jack's and Jill's age, respectively. The first sentence says $x = 2(y-(x-y))$, whence $3x = 4y$. The second sentence then states $((x - y) + y) + ((x - y) + x) = 90$, implying $3x - y = 90$. Therefore $x = 40$ and $y = 30$.

Problem 16J / 6S. Positive integers $a_1, a_2, a_3, \ldots$ form an arithmetic progression. If $a_1 = 10$ and $a_{a_2} = 100$, find $a_{a_{a_3}}$.

Result. 820

Solution outline. It is well known that the k-th term of an arithmetic progression can be written as $a_k = a_1 + (k - 1)d = 10 + (k - 1)d$. It follows that $a_2 = 10 + d$ and thus

$$a_{a_2} = a_{10+d} = 10 + (9 + d)d = 10 + 9d + d^2.$$

From the statement we know that $a_{a_2} = 100$ and hence $100 = 10 + 9d + d^2$. The only positive solution of this quadratic equation is $d = 6$. We infer that $a_k = 4 + 6k$ and the rest is a straightforward computation: $a_3 = 4 + 18 = 22$, $a_{a_3} = a_{22} = 4 + 132 = 136$, and finally $a_{a_{a_3}} = a_{136} = 4 + 816 = 820$.

Problem 17J / 7S. Natali's favorite number has the following properties:

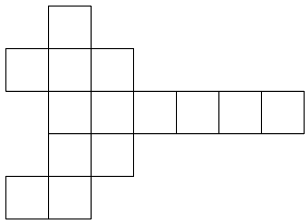
- it has 8 digits;
- its digits are pairwise distinct and decreasing when read from left to right; and
- it is divisible by 180.

What is Natali's favorite number?

Result. 97654320

Solution outline. Natali's number is divisible by 20, so its units digit is 0 and its tens digit is even. However, it cannot be 4, 6 or 8, because then there would not be enough digits to complete an 8 digit number (in the light of the second condition), so it is 2. Further, we decide which of the digits $3, 4, \ldots, 9$ will be missing to ensure the divisibility by 9. Since $0 + 2 + 3 + 4 + \cdots + 9 = 44$, we omit the number 8. Finally, using the second condition, we conclude that Natali's favourite number is 97654320.

Problem 18J / 8S. Kika likes to build models of three-dimensional objects from square ruled paper. Last time she scissored out a shape as shown in the figure below. Then she glued it together in such a way that no two squares were overlapping, there were no holes in the surface of the resultant object and it had nonzero volume. How many vertices did this object have? Note that by a vertex we mean a vertex of the three-dimensional object, not a lattice point on the paper.

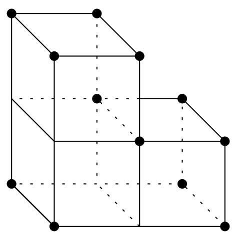


Result. 12

Solution outline. If we look at the resulting object from the front, we see the same number of squares as if we look from behind, similarly for other sides. Denote by a, b, and c the number of squares seen from the front, the left, and above, respectively.

Assume that there are no unseen squares, then we get $2(a+b+c) = 14$ (there are 14 squares altogether). W.l.o.g. we may assume $a \leq b \leq c$. This leaves us with four possible triples (a, b, c), namely $(1, 1, 5)$, $(1, 2, 4)$, $(1, 3, 3)$, and $(2, 2, 3)$. As the inequality $c \leq ab$ must hold, only two possibilities remain: a block with dimensions $1 \times 3 \times 3$ and an "L"-shape, of which only the latter can be constructed from the given shape.

If there was an unseen square, there would have to be at least five squares visible from one side, which we have already ruled out.

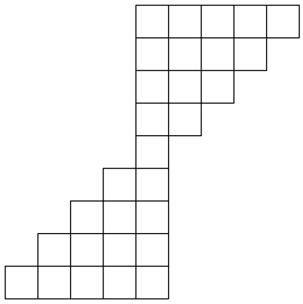


Problem 19J / 9S. Find all pairs of positive integers a, b such that $ab = \gcd(a, b) + \operatorname{lcm}(a, b)$, where $\gcd(a, b)$ and $\operatorname{lcm}(a, b)$ are the greatest common divisor and the least common multiple of a and b, respectively.

Result. $(2, 2)$

Solution outline. Let $d = \gcd(a, b)$. By considering the prime factorizations of a and b, we can easily establish the (general) relation $\operatorname{lcm}(a, b) = ab/d$, hence $ab = d + ab/d$ or $(d - 1)ab = d^2$. As $a, b \geq d$ and $d > 0$, we infer that $a = b = d$ and $d - 1 = 1$, thus $(a, b) = (2, 2)$.

Problem 20J / 10S. There are 29 unit squares in the diagram below. A frog starts in one of the five (unit) squares in the bottom row. Each second, it jumps either to the square directly above its current position (if such a square exists), or to the square that is one above and one to the right from its current square (if such a square exists). The frog jumps every second until it reaches the top. How many distinct paths from the bottom row to the top row can the frog take?



Result. 256

Solution outline. Every path of the frog has to go through the square in the middle. On its way from this square the frog decides four times whether to jump directly up or diagonally, thus the number of ways from the middle up is $2^4 = 16$. The number of ways from the bottom row to the middle is the same as the number of ways from the middle downwards using the opposite moves (because of the symmetry of the diagram), so it is 16 as well. Any two such subpaths may be joined in the middle square to form a full path, so the total number is $16^2 = 256$.

Problem 21J / 11S. Palo is walking along the diagonals of a regular octagon. His walk begins in a fixed vertex of the octagon, and should continue along all the diagonals in such a way that he walks along each diagonal exactly once. How many different walks can Palo take?

Result. 0

Solution outline. Whenever Palo enters a vertex other than the first and the last one of his walk, he has to leave it along a previously unused diagonal. However, there is an odd number of diagonals incident at every vertex (namely five), therefore he cannot use all the diagonals incident to some vertex. We conclude that there is no such walk.

Problem 22J / 12S. Consider a 2014×2014 grid of unit squares with the lower left corner in the point $(0, 0)$ and upper right corner in $(2014, 2014)$. A line p passes through the points $(0, 0)$ and $(2014, 2019)$. How many squares of the grid does p cross?

Note: A line *crosses* a square if they have at least two points in common. For example, the diagonal crosses 2014 squares of the grid.

Result. 4023

Solution outline. As p passes through the points $(0, 0)$ and $(2014, 2019)$, each of its points (x, y) satisfies $x/y = 2014/2019$. Since 2014 and 2019 are coprime, the fraction $2014/2019$ is in its lowest terms and p does not pass through any vertex of a square in the grid other than $(0, 0)$. Thus, p crosses a new square with every intersected inner grid segment. "On its way" from $(0, 0)$ to $(2014, 2019)$, p intersects 2013 such horizontal and $\lfloor 2014^2/2019 \rfloor = 2009$ vertical segments. If we add the corner square in which p "starts", we get $1 + 2013 + 2009 = 4023$ crossed squares in total.

Problem 23J / 13S. A convex n-gon has one interior angle of an arbitrary measure and all the remaining $n - 1$ angles have a measure of $150°$. What are the possible values of n? List all possibilities.

Result. 8, 9, 10, 11, 12

Solution outline. Every n-gon can be cut into $n - 2$ triangles, hence the sum of interior angles is $(n - 2) \cdot 180°$. Thus we have $(n - 2) \cdot 180° = (n - 1) \cdot 150° + x$ or $n = x/30° + 7$, where $0° < x < 180°$ is an arbitrary convex angle. We conclude that $8 \leq n \leq 12$.

Problem 24J / 14S. If the sides a, b, c of a triangle satisfy

$$\frac{3}{a+b+c} = \frac{1}{a+b} + \frac{1}{a+c},$$

what is the angle between the sides b and c?

Result. $60°$

Solution outline. After multiplying the whole expression appropriately to get rid of fractions we obtain

$$3(a^2 + ab + ac + bc) = 2a^2 + b^2 + c^2 + 3(ab + ac) + 2bc,$$

so $a^2 = b^2 + c^2 - bc$. Comparison with the law of cosines ($a^2 = b^2 + c^2 - 2bc \cos \alpha$) yields $\cos \alpha = \frac{1}{2}$ and $\alpha = 60°$.

Problem 25J / 15S. Find all integers between 1 and 200, whose distinct prime divisors sum up to 16. (For example, the sum of the distinct prime divisors of 12 is $2 + 3 = 5$.)

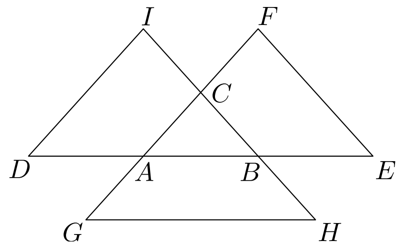
Result. 66, 132, 198, 55, 39, 117

Solution outline. The number 16 can be written as a sum of distinct primes in the following ways:

$$16 = 2 + 3 + 11 = 3 + 13 = 5 + 11$$

The first decomposition corresponds to the numbers of the form $2^a 3^b 11^c$ with a, b, c positive integers; only $2 \cdot 3 \cdot 11 = 66$, $2^2 \cdot 3 \cdot 11 = 132$ and $2 \cdot 3^2 \cdot 11 = 198$ are less or equal 200. The second case yields $3 \cdot 13 = 39$, $3^2 \cdot 13 = 117$ and from the last one we get $5 \cdot 11 = 55$.

Problem 26J / 16S. The lengths of line segments in the figure satisfy $DA = AB = BE$, $GA = AC = CF$ and $IC = CB = BH$. Moreover, $EF = DI = 5$ and $GH = 6$. What is the area of triangle ABC?



Result. 3

Solution outline. The line segment AC joins the midpoints of DB and BI. Therefore its length must be $AC = DI/2 = 2.5$. Similarly, $CB = EF/2 = 2.5$ and $AB = GH/2 = 3$. We see that ABC is an isosceles triangle and its area can thus be computed by the Pythagorean theorem as

$$\sqrt{AC^2 - (AB/2)^2} \cdot AB/2 = \sqrt{6.25 - 2.25} \cdot 1.5 = 3.$$

Problem 27J / 17S. We call a prime p *strong* if one of the following conditions holds:

- p is a one-digit prime; or
- if we remove its first digit, we obtain another strong prime, and the same holds for the last digit.

For example, 37 is a strong prime, since by removing its first digit we get 7 and by removing its last digit we get 3 and both 3 and 7 are strong primes. Find all strong primes.

Result. 2, 3, 5, 7, 23, 37, 53, 73, 373

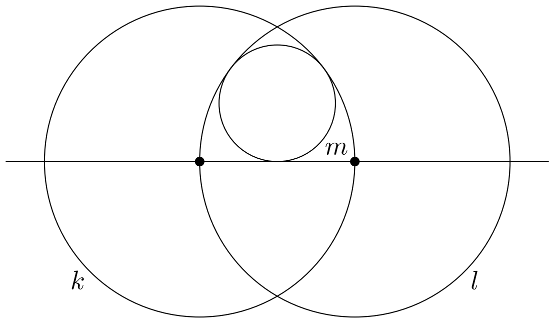
Solution outline. One-digit strong primes (1SP) are 2, 3, 5 and 7.

Two-digit strong primes (2SP) are prime numbers obtained by joining two 1SP together, namely 23, 37, 53, and 73.

Let us now find the three-digit strong primes (3SP): By removing the first digit we get a 2SP and similarly by removing the last digit. Therefore, we seek two 2SP such that the second digit of one is the first digit of the other. Out of the possible candidates 237, 537, 737 and 373, only 373 is a prime.

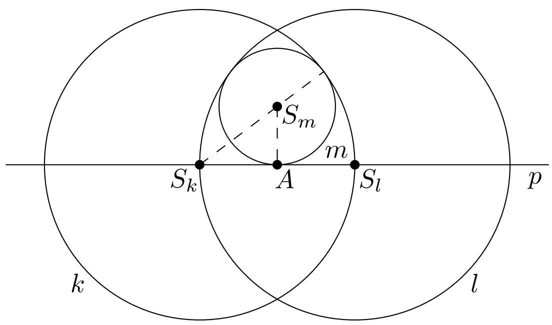
Finally, we focus on the four-digit strong primes (4SP). After removing both the first and the last digit we need to obtain 3SP, i.e. 373, which is clearly impossible. Hence 4SP do not exist and neither do strong primes with more than 4 digits. To sum up, the only strong primes are 2, 3, 5, 7, 23, 37, 53, 73, and 373.

Problem 28J / 18S. Let k and l be two circles of radius 16 such that each of them passes through the centre of the other. A circle m is internally tangent to both k and l and also tangent to the line p passing through their centres. Find the radius of the circle m.



Result. 6

Solution outline. Let S_k, S_l and S_m be the centres of the circles k, l, and m respectively. Denote by r the radius of the circle m. We know S_k, S_m and the point where m touches k lie on one line, whence $S_kS_m = 16 - r$. Let A be the point where the circle m touches the line p. By symmetry we have $S_kA = S_lA = 16/2 = 8$. Clearly, $S_mA = r$ and the line S_mA is perpendicular to p, so we can use the Pythagorean theorem in $\triangle S_kAS_m$, yielding the equation $8^2 + r^2 = (16 - r)^2$ with the unique solution $r = 6$.

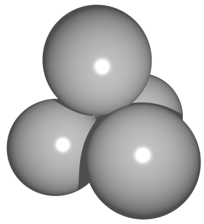


Problem 29J / 19S. How many six-digit positive integers satisfy that each of their digits occurs the same number of times as is the value of the digit? An example of such an integer is 133232.

Result. 82

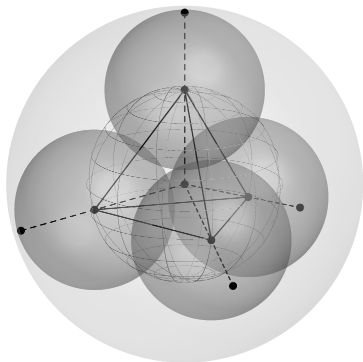
Solution outline. Firstly observe that the sum of distinct digits of any such number is 6. As 0 cannot occur among these numbers, we are looking for partitions of 6 into distinct positive integers; these are 6 alone, $1 + 5$, $1 + 2 + 3$ and $2 + 4$. There are $6 \cdot \binom{5}{2} = 60$ numbers with digits 1, 2, and 3, since there are six possible positions for digit 1 and then we choose two positions for digit 2 out of 5 remaining. Using similar elementary combinatorics, we compute that the partitions 6, $1 + 5$ and $2 + 4$ provide 1, 6, and $\binom{6}{2} = 15$ valid numbers. Therefore, there are $60 + 1 + 6 + 15 = 82$ numbers satisfying the given condition.

Problem 30J / 20S. True to his reputation as a cool guy, E.T. glued together 4 balls of radius 1 so that they were pairwise tangent. What is the radius of the smallest sphere containing E.T.'s 4 balls?



Result. $1+\frac{\sqrt{6}}{2}$

Solution outline. When we decrease the radius of the containing sphere by 1, we obtain the circumscribed sphere C of the regular tetrahedron T formed by the centres of the given balls. Note that edges of T are of the length 2. Using the Pythagorean theorem, we discover that faces of T have altitudes of the length $\sqrt{4-1}=\sqrt{3}$. Now we can calculate the length of the altitudes of T, once again by means of the Pythagorean theorem, obtaining $\sqrt{4-(\frac{2}{3}\sqrt{3})^2}=\frac{2}{3}\sqrt{6}$. It is well known (or we may employ the Pythagorean theorem for the third time) that the centre of C divides the altitude with the ratio $1:3$, yielding the radius of C as $\frac{\sqrt{6}}{2}$ and the desired value $1+\frac{\sqrt{6}}{2}$.



Problem 31J / 21S. Determine the number of pairs of positive integers (x,y) with $y<x\leq 100$ such that the numbers x^2-y^2 and x^3-y^3 are coprime.

Result. 99

Solution outline. Recall $x^2-y^2=(x-y)(x+y)$ and $x^3-y^3=(x-y)(x^2+xy+y^2)$. In order to achieve $\gcd((x-y)(x+y),(x-y)(x^2+xy+y^2))=1$, we need $x-y=1$, i.e. $x=y+1$. Substituting for y, we search for x such that $\gcd(2x-1,3x^2-3x+1)=1$. After subtracting $(2x-1)(x-1)$ from $3x^2-3x+1$, we get an equivalent condition $\gcd(2x-1,x^2)=1$. Suppose there exists a common prime divisor of $2x-1$ and x^2 and denote it p. Then $p\mid x$ and $p\mid 2x-1$, whence $p\mid x-1$. This contradicts $p>1$, so the assumption is false and $\gcd(2x-1,3x^2-3x+1)=1$ for any $x\geq 1$. Given that we have shown $x=y+1$, the desired pairs are $(2,1),(3,2),\ldots,(100,99)$. All in all there are 99 of them.

Problem 32J / 22S. Consider a number that starts with $122333444455555\ldots$ and continues in such a way that we write each positive integer as many times as its value indicates. We stop after writing 2014 digits. What is the last digit of this number?

Result. 4

Solution outline. One-digit numbers require $1+2+\cdots+9=45$ digits, two-digit numbers $2\cdot(10+11+\cdots+99)=2\cdot(\frac{99\cdot 100}{2}-\frac{9\cdot 10}{2})=9810$. Hence we reach 2014 digits while writing a two-digit number. Observe that we are asking for a digit in an even position, and considering one-digit numbers take up an odd length, we only need to know the value of the digit on tens place for the number that occupies the 2014th position. The last written number is the least n such that $45+2\cdot(10+11+\cdots+n)>2014$, which yields a rough estimate $40<n<50$, and so the last digit is 4.

Problem 33J / 23S. What is the smallest positive integer N for which the equation $(x^2-1)(y^2-1) = N$ has at least two distinct integer solutions (x, y) such that $0 < x \le y$?

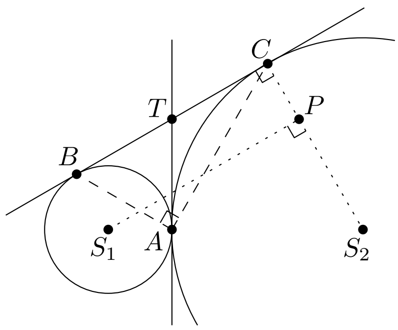
Result. 360

Solution outline. Let us list the first few values that can be obtained by $(t^2 - 1)$ for integer t: 0, 3, 8, 15, 24, 35, 48, 63, 80, 99, 120, etc. We want to find two pairs with the same minimal product. Divine revelation yields $3 \cdot 120 = 15 \cdot 24 = 360$, which happens to be the smallest possibility actually: Should one of the factors be greater than 120, the product would exceed 360. Therefore we may confine ourselves to the factors listed above. Since at most one factor may be 3, we may concentrate only on the products without this factor smaller than 360, i.e. $8 \cdot 8$, $8 \cdot 15$, $8 \cdot 24$, $8 \cdot 35$, and $15 \cdot 15$. These products, however, do not allow for an alternative factorization. Hence the smallest N really is 360.

Problem 34J / 24S. Let two circles of radii 1 and 3 be tangent at point A and tangent to a common straight line (not passing through A) at points B and C. Find $AB^2 + BC^2 + AC^2$.

Result. 24

Solution outline. Let S_1 be the centre of the smaller circle, S_2 the centre of the larger one and T the point where the line BC intersects the common tangent passing through A. Note that $TC = TA = TB$ (tangent lines). Thales' theorem then implies that the triangle ABC is right-angled and we obtain $AB^2 + AC^2 = BC^2$. Let us now translate the line segment BC so that B is moved onto S_1 and denote by P the translated point C. Noting $CP = S_1B = 1$, the Pythagorean theorem entails $BC^2 = S_1P^2 = (S_1A + S_2A)^2 - (S_2C - CP)^2 = 12$, yielding the solution 24.



Problem 35J / 25S. Find the largest prime p such that $p^p \mid 2014!$.

Result. 43

Solution outline. Let q be a prime. Then factors from 2014! divisible by q are $q, 2q, \ldots, \lfloor 2014/q \rfloor q$. Denoting $m = \lfloor 2014/q \rfloor$, we have $q^m \mid 2014!$. If $q > m$ then $q^2 > mq$ and $q^{m+1} \nmid 2014!$, implying $q^q \nmid 2014!$, which is undesirable. Conversely, if $q \le m$, then clearly $q^q \mid 2014!$. Hence p must be the largest prime satisfying $p \le \lfloor 2014/p \rfloor$, i.e. $p^2 \le 2014$, which results in $p = 43$.

Problem 36J / 26S. Consider an array of 2 columns and 2014 rows. Using 3 different colours, we paint each cell of the array with a colour so that if the cells share a wall, they are of different colours. How many different colourings are there?

Result. $6 \cdot 3^{2013}$

Solution outline. First we colour the bottom row arbitrarily, yielding $3 \cdot 2 = 6$ possibilities. If a row is coloured (a, b) then the next one can be (b, a), (b, c) or (c, a). In other words, independently of choosing a colouring for a row, we have exactly three possibilities of how to colour the following one. That gives us $6 \cdot 3^{2013}$ different colourings.

Problem 37J / 27S. Find the smallest positive integer m such that $5m$ is the fifth power of an integer, $6m$ is the sixth power of an integer and $7m$ is the seventh power of an integer.

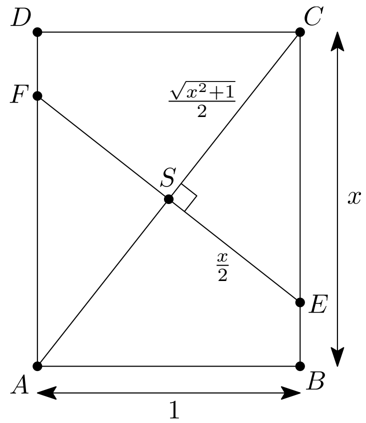
Result. $5^{84}6^{35}7^{90}$

Solution outline. Let us write $m = 5^a 6^b 7^c d$ where d is divisible by neither 5, 6 nor 7. In order for $5m$ to be a fifth power, we need 5 to divide $a+1$, b and c. Likewise $6 \mid a, b+1, c$ and $7 \mid a, b, c+1$. The number d has to be a fifth, sixth and seventh power at the same time; we are looking for the smallest integer, so we let $d = 1$. The number a is divisible by 42 and $a+1$ by 5, therefore the smallest suitable a is 84. We find $b = 35$ and $c = 90$ analogously. The smallest suitable m turns out to be $5^{84}6^{35}7^{90}$.

Problem 38J / 28S. We fold a rectangular piece of paper in such a way that two diagonally opposite vertices are touching, thus creating a *fold*, i.e. a line segment across the paper. After folding, the length of the fold line has the same length as the longer side of the rectangle. What is the ratio of the length of a longer side to the length of a shorter side of the rectangle?

Result. $\sqrt{\frac{1+\sqrt{5}}{2}}$

Solution outline. Label important points as in the picture. Since similar rectangles yield the same result, we may w.l.o.g. assume $AB = 1$. The question is now to find $x = BC$. First note that $EF \perp AC$, as EF is the axis of symmetry of AC. We can see that $ES = FS = \frac{x}{2}$ by symmetry. The Pythagorean theorem implies $CS = \frac{\sqrt{x^2+1}}{2}$. Triangles ABC and ESC are similar, hence we infer $\frac{x}{1} = \frac{\sqrt{x^2+1}}{x}$. This relation yields $x^4 - x^2 - 1 = 0$ which, after substituting $z = x^2$, produces a unique non-negative solution $x = \sqrt{\frac{1+\sqrt{5}}{2}}$.



Problem 39J / 29S. We have 20 marbles, each of which is either yellow, blue, green, or red. Assuming marbles of the same colour are indistinguishable, how many marbles can be blue at most if the number of ways we can arrange the marbles into a straight line at most is 1140?

Result. 17

Solution outline. Let b, y, g, and r denote the number of blue, yellow, green, and red marbles, respectively. We have $\binom{20}{r}$ different ways of placing the red marbles. There remain $20 - r$ positions for the green marbles, hence $\binom{20-r}{g}$ different ways of placing them, having placed the red ones. Similarly, the yellow marbles can be placed in $\binom{20-r-g}{y}$ different ways after fixing the red and green marbles. The remaining blue marbles just fill in the void. So we seek the smallest $y + g + r$ such that

$$\binom{20}{r}\binom{20-r}{g}\binom{20-r-g}{y} = 1140.$$

We can manipulate the expression to obtain

$$1140 = \frac{20(20-1)\cdots(b+1)}{r!g!y!}.$$

If $b \geq 18$, the numerator is at most $20 \cdot 19 = 380$, so the number of ways is less than 1140. For $b = 17$ the number 1140 can be attained via setting $y = 3$, $g = 0$, $r = 0$.

Problem 40J / 30S. Find all integers $n \geq 3$ such that the regular n-gon can be divided into at least two regular polygons.

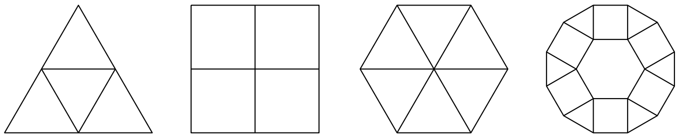
Result. 3, 4, 6, 12

Solution outline. Firstly observe that an equilateral triangle can be divided into four equilateral triangles by its midlines, a square can be divided into four squares and a regular hexagon into six triangles.

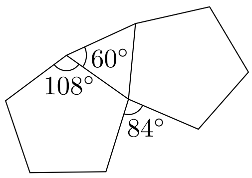
Consider a regular n-gon with $n \neq 3, 4, 6$. The internal angles of regular polygons with 3, 4, 5, and 6 vertices are $60°$, $90°$, $108°$, and $120°$, respectively. We have to "fill" the internal angle of the n-gon either using a smaller n-gon or two polygons with at most five vertices. However, the former option leaves us with an acute angle different from $60°$, which cannot be filled.

Therefore, the internal angles of the n-gon have to be filled either with a triangle and a square or with a triangle and a pentagon. The corresponding interior angles are $150°$ and $168°$, i.e. n is 12 or 30.

It is indeed possible to divide the regular dodecagon: We alternately put squares and triangles to the sides of the polygon and a hexagon to the middle.



On the other hand, such a division of a triacontagon does not exist; it would need to have a pentagon adjoined to every other side. Two such neighbouring pentagons would meet in a vertex with one angle equal to $60°$, thus the angle on the other side would be $84°$, which cannot be filled.



Problem 41J / 31S. A rook is standing in the bottom left corner of a 5×5 chessboard. In how many ways can the rook reach the top right corner of the board, provided that it is limited to move only up or to the right? We consider two paths to be distinct if the sequences of the visited squares are different.

Result. 838

Solution outline. For each square we compute the number of ways to reach it. There is only one way for the bottom left corner (empty sequence of moves). For every other square S we take the following approach: Consider a sequence of moves ending in S and exclude its last move; before this move, the rook was in a square T either below or to the left of S. On the other hand, whenever the rook gets to a square T below or to the left of S, by adding one move we obtain a way to reach S. Therefore if we know the numbers of ways to get to all such squares T, it suffices to sum up all these ways.

Thus we employ the following procedure: Firstly, we write 1 to the bottom left corner of the board and then we repeatedly apply the rule "Find a square with all squares below and to the left filled in and fill it with the sum of these values." The sought number is in the top right corner.

8	28	94	289	838
4	12	37	106	289
2	5	14	37	94
1	2	5	12	28
1	1	2	4	8

Problem 42J / 32S. What is the digit in the place of hundreds in 11^{2014}?

Result. 2

Solution outline. The binomial theorem entails

$$11^{2014} = (1+10)^{2014} = \binom{2014}{0} \cdot 1 + \binom{2014}{1} \cdot 10 + \binom{2014}{2} \cdot 100 + 1000 \cdot \left(\binom{2014}{3} + \cdots \right).$$

We need to compute this expression modulo 1000 and to take the first digit, so we may drop the terms grouped in the last brackets and proceed as

$$11^{2014} \equiv 1 + 2014 \cdot 10 + 1007 \cdot 2013 \cdot 100 \equiv 1 + 140 + 7 \cdot 3 \cdot 100 \equiv 241 \pmod{1000}.$$

Problem 43J / 33S. A grasshopper is jumping on vertices of an equilateral triangle. Whenever it sits on a vertex, it randomly chooses one of the other two vertices as the destination for its next jump. What is the probability that it returns to the starting vertex with its tenth jump?

Result. $\frac{171}{512}$

Solution outline. Denote by A the starting vertex and by a_i the probability of the grasshopper being in the vertex A after i jumps (so $a_0 = 1$). We can find a recurrence relation for a_{i+1} provided that a_i is given: If the $(i+1)$-th jump of the grasshopper ended in A, then the previous jump did not and the grasshopper had to choose vertex A out of the two possible ones, so

$$a_{i+1} = \frac{1}{2}(1 - a_i)$$

or, equivalently,

$$a_{i+1} - \frac{1}{3} = -\frac{1}{2}\left(a_i - \frac{1}{3}\right).$$

As $a_0 = 1/3 + 2/3$, we infer that

$$a_i = \frac{1}{3} + \frac{2}{3} \cdot \left(-\frac{1}{2}\right)^i$$

and thus

$$a_{10} = \frac{1}{3} + \frac{1}{3 \cdot 512} = \frac{171}{512}.$$

Problem 44J / 34S. Parsley is a lumberjack. His work is strictly divided into minutes, in which he may choose one of the following actions:

- He cuts down n trees, where n is his current power. This wears him out, so his power decreases by one.
- He takes a rest and restores one point of power.

What is the maximal number of trees he may cut down in 60 minutes, if his initial power is 100?

Result. 4293

Solution outline. If Parsley spent a minute resting and in the following minute he cut trees, he would cut one tree less than if he firstly cut and then rested, but in neither case does his power change. Therefore, in order to maximize the number of trees cut, he begins with r minutes of rest and proceeds with $60 - r$ minutes of cutting for some r. The total number of trees cut is then

$$(r+100)+(r+99)+\cdots+(r+100-(60-r-1)) = \frac{(r+100+2r+41)}{2} \cdot (60-r) = -\frac{3}{2}((r-6.5)^2 - 2862.25).$$

The maximum is attained if the term $(r-6.5)^2$ is lowest possible, i.e. for $r = 6$ or $r = 7$. In this case Parsley cuts $\frac{3}{2} \cdot 2862 = 4293$ trees.

Problem 45J / 35S. Denote by $\sqrt{\mathbb{N}}$ the set $\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \dots\}$, i.e. the set of all square roots of positive integers. Let S be the set of real numbers a for which both $a \in \sqrt{\mathbb{N}}$ and $36/a \in \sqrt{\mathbb{N}}$ hold. What is the product of the elements of the set S?

Result. 6^{25}

Solution outline. If $a = \sqrt{n_{a,1}}$ and $36/a = \sqrt{n_{a,2}}$ for some $n_{a,1}, n_{a,2} \in \mathbb{N}$, it follows $n_{a,1}n_{a,2} = 36^2$. This observation lets us compute the desired product as follows:

$$\prod_{a\in S} a = \sqrt{\prod_{a\in S} n_{a,1}} = \sqrt{\prod_{n|36^2} n} = \sqrt{\prod_{0\le k_2\le 4}\prod_{0\le k_1\le 4} 2^{k_1}3^{k_2}} = \sqrt{\prod_{0\le k_2\le 4} 2^{10}3^{k_2}} = \sqrt{2^{50}3^{50}} = 6^{25}.$$

Problem 46J / 36S. What is the probability that the product of 2014 randomly chosen digits is a multiple of 10?

Result. $1 - \left(\frac{5}{10}\right)^{2014} - \left(\frac{8}{10}\right)^{2014} + \left(\frac{4}{10}\right)^{2014}$

Solution outline. We seek the probability that the product is divisible by both 2 and 5. Let us compute the probability of the complementary event: the probability of the product not being divisible by 2 is $a = (5/10)^{2014}$ (all the factors have to be odd). Similarly, for the divisibility by 5 we have $b = (8/10)^{2014}$ (there may be no 0 or 5), and the probability of not being divisible neither by 2 nor 5 is $c = (4/10)^{2014}$. The sought probability is $1 - a - b + c$, since we have subtracted the probability of non-divisibility by 2 and 5 twice and thus we have to add it back.

Problem 47J / 37S. Evaluate the expression

$$\left\lfloor 2014\sqrt[3]{-2014}\right\rfloor + \left\lfloor 2014\sqrt[3]{-2013}\right\rfloor + \dots + \left\lfloor 2014\sqrt[3]{2014}\right\rfloor.$$

Note: The symbol $\lfloor x\rfloor$ denotes the *integral part* of x, e.g. the greatest integer not exceeding x.

Result. -2002

Solution outline. We rewrite the expression as

$$\begin{aligned}&\left\lfloor 2014\sqrt[3]{0}\right\rfloor + \left(\left\lfloor 2014\sqrt[3]{1}\right\rfloor + \left\lfloor 2014\sqrt[3]{-1}\right\rfloor\right) + \dots + \left(\left\lfloor 2014\sqrt[3]{2014}\right\rfloor + \left\lfloor 2014\sqrt[3]{-2014}\right\rfloor\right) = \\ &\quad = \left(\left\lfloor 2014\sqrt[3]{1}\right\rfloor - \left\lceil 2014\sqrt[3]{1}\right\rceil\right) + \left(\left\lfloor 2014\sqrt[3]{2}\right\rfloor - \left\lceil 2014\sqrt[3]{2}\right\rceil\right) + \dots + \left(\left\lfloor 2014\sqrt[3]{2014}\right\rfloor - \left\lceil 2014\sqrt[3]{2014}\right\rceil\right),\end{aligned}$$

where $\lceil x\rceil$ denotes the smallest integer greater or equal to x. Observe that if x is an integer, then $\lfloor x\rfloor - \lceil x\rceil$ is equal to zero; otherwise it is -1. As the cube root of an integer is either an integer or an irrational number, the sum in question is minus the number of numbers between 1 and 2014 (inclusive) which are not a cube. Since $12^3 = 1728 < 2014$ and $13^3 = 2197 > 2014$, there are exactly 12 cubes between 1 and 2014, so the desired value is $-(2014 - 12) = -2002$.

Problem 48J / 38S. Boris counted the sum $1 + 2 + \dots + 2012$. However, he forgot to add some numbers and got a number divisible by 2011. Ann counted the sum $A = 1 + 2 + \dots + 2013$. However, she missed the same numbers as Boris and got a number N divisible by 2014. What is the ratio N/A?

Result. $2/3$

Solution outline. Boris's number is $N - 2013$, so

$$N \equiv 2013 \equiv 2 \pmod{2011}$$

and $N \equiv 0 \pmod{2014}$. We may similarly proceed with the number $2A$:

$$2A = 2013 \cdot 2014 \equiv 0 \pmod{2014}$$

and $2A \equiv 2 \cdot 3 = 6 \pmod{2011}$. Since $2A$ is divisible by 3 (as it is divisible by 2013), but neither of 2011 and 2014 is, we may divide the congruences for A by 3 and find that the number $2A/3$ satisfies precisely the conditions for N. On the other hand, no other such N is admissible, since the distance of the next closest numbers satisfying the congruences for N is $2011 \cdot 2014 > A$. Thus $N/A = 2/3$.

Problem 49J / 39S. A game is played with 16 cards laid in a row. Each card has one side black and the other side red. A move consists of taking a consecutive sequence of cards (possibly only containing 1 card) in which the leftmost card has its black side face-up and the rest of the cards have red side face-up, and flipping all of these cards over. The game ends when a move can no longer be made. What is the maximum possible number of moves that can be made before the game ends?

Result. $2^{16} - 1$

Solution outline. Imagine the row of 16 cards as a binary number where a black side face-up corresponds to 0 and a red one to 1. A move in our game then corresponds to binary addition of some number to thus interpreted row of cards so that the sum would always remain less than 2^{16}. Therefore the game cannot run forever and we have an upper bound for the number of moves $2^{16} - 1$. This number of moves can actually be attained: consider the initial state consisting of all zeroes (i.e. all cards are black side face-up) and perform moves corresponding to adding 1 in every move.

Problem 50J / 40S. We have a right-angled triangle with sides of integral lengths such that one of the catheti is 2014^{14} in length. How many such triangles exist up to congruence?

Result. $\frac{27\cdot 29^2-1}{2}$

Solution outline. Denote the length of the second cathetus b and the hypotenuse c. The Pythagorean theorem implies $(c-b)(c+b) = 2014^{28}$. Since all the lengths are integers, the question is in how many ways we can write 2014^{28} as a product of two distinct, even divisors: the numbers $c-b$ and $c+b$ have the same parity, hence both must be even. The possibility $c-b = c+b$ is ruled out for it implies $b = 0$. We have $2014^{28} = 2^{28} \cdot 19^{28} \cdot 53^{28}$, so there are $27 \cdot 29 \cdot 29 - 1$ pairs of divisors satisfying the conditions. Given that we have to take into account also $c-b < c+b$, we get $\frac{27\cdot 29^2-1}{2}$ possible triangles.

Problem 51J / 41S. What is the smallest positive integer that cannot be written as a sum of 11 or fewer (not necessarily distinct) factorials?

Result. 359

Solution outline. Observe that every $n \in \mathbb{N}$ can be uniquely decomposed as $n = \sum_{k=1}^{N} a_k k!$ for some N dependent on n and $a_k \in \mathbb{N}$ such that $a_k \leq k$ since $(k+1)k! = (k+1)!$. For instance $43 = 1 + 3 \cdot 3! + 4!$. The question asked can thus be reformulated as finding the smallest n for which $\sum_{k=1}^{N} a_k = 12$. This question can now easily be answered as the corresponding a_k will be $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 4$ and $a_5 = 2$ so that $n = \sum_{k=1}^{5} a_k k! = 359$.

Problem 52J / 42S. In a chess tournament, each participant plays against every other participant, and there are no draws. Call a group of four chess players *ordered* if there is a clear winner and a clear loser, that is, one person who beat the other three and one person who lost to the other three. Find the smallest integer n for which any chess tournament with n people has a group of four chess players that is ordered.

Result. 8

Solution outline. Note first that for 8 participants, the average number of victories of a player is 3.5, and hence there must be a player v who won (at least) 4 times, say against players a_1, a_2, a_3 and a_4. Similarly, one of these four players must have won against at least two other of them, so without loss of generality assume that a_1 defeated a_2 and a_3. Finally, somebody won the match a_2 against a_3, we can without loss of generality assume it was a_2. Observe that the group $\{v, a_1, a_2, a_3\}$ is ordered. Hence we see $n \leq 8$.

Actually $n = 8$, because a smaller number of participants permits a tournament without an ordered group of 4 players. To prove this, assume without loss of generality there are 7 players $a_0, \ldots, a_6$. For every $0 \leq i \leq 6$, consider the player a_i won the match with a_j where $j = (i+1) \bmod 7$, $(i+2) \bmod 7$, and $(i+4) \bmod 7$. The only possible candidates for an ordered group of size 4 are quadruples of a player and the three beaten by him/her. However, such groups are not ordered.

Problem 53J / 43S. I chose two numbers from the set $\{1, 2, \ldots, 9\}$. Then I told Peter their product and Dominic their sum. The following conversation ensued:

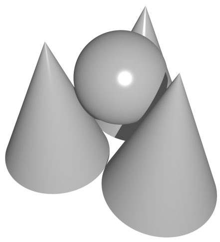
- Peter: "I don't know the numbers."
- Dominic: "I don't know the numbers."
- Peter: "I don't know the numbers."
- Dominic: "I don't know the numbers."
- Peter: "I don't know the numbers."
- Dominic: "I don't know the numbers."
- Peter: "I don't know the numbers."
- Dominic: "I don't know the numbers."
- Peter: "Now I know the numbers."

What numbers did I choose?

Result. 2, 8

Solution outline. Given that there must be a solution, both players can be expected to behave perfectly rationally. They write sums and products of all pairs from $\{1, \ldots, 9\}$. If Peter saw that the number he had been told was there only once, he would have known the original numbers immediately. But he could not infer a clear answer. Therefore Dominic might ignore all pairs of numbers that have a unique product. Then if Dominic had a unique sum, he would have known the numbers, but he did not, therefore Peter could cross out all pairs with a unique sum. And in this manner we carry on.

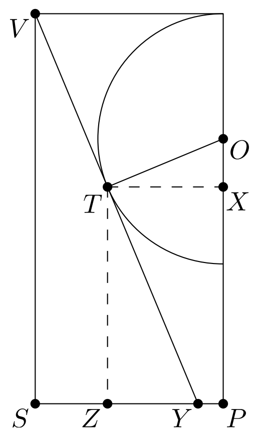
Problem 54J / 44S. Three identical cones are placed in the space in such a way that their bases are pairwise incident and all three bases completely lie in a single plane. We place a sphere between the cones so that the top of the sphere is in the same height as the vertices of the cones. What is the radius of the ball if each cone has base of radius 50 cm and height 120 cm?



Result. $\frac{200\sqrt{3}}{9}$

Solution outline. Denote by r the radius of the sphere and O the center of the sphere. Select one of the cones and denote the center of its base as S, its vertex as V and its tangent point with the sphere as T. Let b be the plane in which bases of the cones lie. Let P be the orthogonal projection of O to b, X be on the line OP so that TX is parallel with b, and Y be the intersection of the line VT and the boundary of the selected cone. Finally, denote Z the orthogonal projection of T to b.

The Pythagorean theorem implies $VY = 130$. As the sphere touches the cone, VY is perpendicular to TO. Hence $\triangle TXO$ is similar to $\triangle VSY$ and therefore $OX = \frac{5}{13}r$ and $TX = \frac{12}{13}r$. Thus $TZ = 120 - \frac{18}{13}r$. From the similarity of $\triangle VYS$ and $\triangle TYZ$, we get $YZ = (120 - \frac{18}{13}r) \cdot \frac{5}{12} = 50 - \frac{15}{26}r$ and consequently $SZ = \frac{15}{26}r$. On the other hand, $PZ = TX = \frac{12}{13}r$. Observe that the base centers of the cones constitute an equilateral triangle, with the side length 100, for which P is the center of mass. So $SP = \frac{100\sqrt{3}}{3}$ and $SZ = \frac{100\sqrt{3}}{3} - \frac{12}{13}r$. With two expressions for SZ, we equate $\frac{100\sqrt{3}}{3} - \frac{12}{13}r = \frac{15}{26}r$, yielding $r = \frac{200\sqrt{3}}{9}$.



Problem 55J / 45S. Imagine a rabbit hutch formed by 7×7 cell grid. In how many ways can we accommodate 8 indistinguishable grumpy rabbits in such a fashion that every two rabbits are at least either 3 columns or 3 rows apart?

Result. 51

Solution outline. Let the columns of the hutch be called A–G and the rows 1–7. We will hold a discussion according to the position of the rabbit closest to the center. If the central cell (D4) is taken, the remaining rabbits can live only along the perimeter. By means of analysing every allowable pattern of such an accommodation and utilizing the rotational symmetry, we obtain there are 36 possibilities then.

Consider now a rabbit dwells in D3. Then we have 2 alternatives, depending on whether D6 or D7 is used. By symmetry we obtain that occupation of D3, D5, C4 or E4 yields 8 possibilities overall. If D2 is taken, we have 2 alternatives not covered by the previous case. Since one of them coincides with taking D6, we observe that occupation of D2, D6, B4 or F4 yields 6 possibilities overall. Next, if C3, B2 or C2 (or symmetries thereof) is occupied, it can be easily verified there is no way how to place the 7 remaining rabbits. Lastly, there is only one way how to place all the rabbits along the perimeter. All in all, we have $36 + 8 + 6 + 1 = 51$ possibilities.

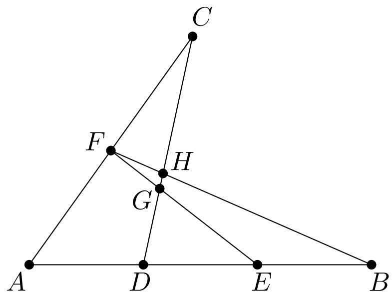
	A	B	C	D	E	F	G
1	●			●			●
2							
3							
4	●			○			●
5							
6							
7	×	×	×	×	×	×	×

	A	B	C	D	E	F	G
1	●						●
2							
3				○			
4	●						●
5							
6				×			
7	●			×			●

	A	B	C	D	E	F	G
1	●						●
2				○			
3							
4	●						●
5				×			
6				×			
7	●			×			●

	A	B	C	D	E	F	G
1	●			○			●
2							
3							
4	●						●
5							
6							
7	●			●			●

Problem 56J / 46S. Given noncollinear points A, B, C, the segment AB is trisected by points D and E. Furthermore, F is the midpoint of the segment AC, and EF and BF intersect CD at G and H, respectively. Compute $[FGH]$ provided that $[DEG] = 18$. Note that by $[XYZ]$ we denote the area of $\triangle XYZ$.

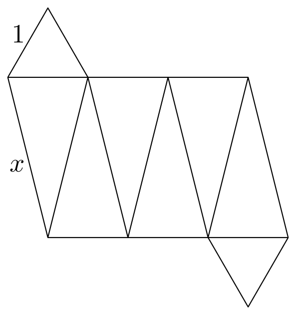


Result. $\frac{9}{5}$

Solution outline. We will solve this problem using mass point geometry. Physical intuition states that masses on opposite sides of a fulcrum balance if and only if the products of the masses and their distances from the fulcrum are equal (in physics-speak, the net torque is zero). If a mass of weight 1 is placed at vertex B and masses of weight 2 are placed at vertices A and C, then $\triangle ABC$ balances on the line BF and also on the line CD. Thus it balances on the point H where these two lines intersect. Replacing the masses at A and C with a single mass of weight 4 at their center of mass F, the triangle still balances at H. Thus $\frac{BH}{HF} = 4$.

Next, consider $\triangle BDF$. Placing masses of weight 1 at the vertices B and D and a mass of weight 4 at F, the triangle balances at G. A similar argument shows that $\frac{EG}{GF} = 2$ and that $\frac{DG}{GH} = 5$. Because $\triangle DEG$ and $\triangle FHG$ have congruent (vertical) angles at G, it follows that $\frac{[DEG]}{[FHG]} = \frac{EG}{FG} \cdot \frac{DG}{HG} = 2 \cdot 5 = 10$. Thus $[FGH] = \frac{[DEG]}{10} = \frac{18}{10} = \frac{9}{5}$.

Problem 57J / 47S. The surface of a solid consists of two equilateral triangles with sides of length 1 and of six isosceles triangles with legs of length x and the base of length 1, as shown below. What is the value of x given that the volume of the object is 6?



Result. $\frac{5\sqrt{39}}{3}$

Solution outline. First consider a regular octahedron of side length 1, that is, the case $x = 1$. To compute its volume, divide it into two square-based pyramids with edges of length 1. Such a pyramid has height $\sqrt{2}/2$, so its volume is

$$\frac{1}{3} \cdot 1^2 \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{6}.$$

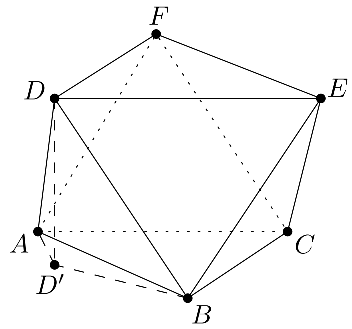
We conclude that the volume of a regular octahedron is $\sqrt{2}/3$.

Note that the given solid can be obtained from the regular octahedron by stretching it appropriately in the direction perpendicular to a pair of faces of the solid; let k be the ratio of this dilation. Since octahedron dilates in one dimension only, its volume increases k times as well. As the volume of the solid is 6, we get $k = 9\sqrt{2}$.

Denote the vertices of the non-stretched triangles A, B, C and D, E, F in such a way that if we project D, E, F to the plane ABC and denote the projections D', E', F', respectively, then $AD'BE'CF'$ is a regular hexagon. From $AB = 1$ we easily establish $AD' = \sqrt{3}/3$ and

$$AD = \sqrt{h^2 + AD'^2} = \sqrt{h^2 + \frac{1}{3}},$$

where h is the height of the solid (based on ABC).



Applying this formula in the case of a regular octahedron ($AD = 1$) yields $h_{\text{reg}} = \sqrt{2/3}$. The height of the stretched octahedron is k times longer, i.e. $h_{\text{str}} = 18/\sqrt{3}$. Substituting to the formula above yields

$$AD = \sqrt{h_{\text{str}}^2 + \frac{1}{3}} = \sqrt{\frac{325}{3}} = \frac{5\sqrt{13}}{\sqrt{3}} = \frac{5\sqrt{39}}{3}.$$