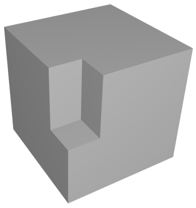
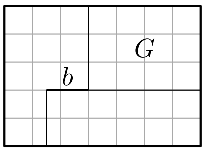
**Problem 1.** Given is a cubical-shaped boulder with an original volume of $216\,\mathrm{m}^3$. What size is the surface of the boulder in $\mathrm{m}^2$ after knocking out a cuboidal block of dimensions $1\,\mathrm{m} \times 1\,\mathrm{m} \times 2\,\mathrm{m}$ as shown in the picture below?



*Result.* 216

*Solution.* Since $6^3 = 216$, the side length of the cube is $6\,\mathrm{m}$. The missing block does not change the surface of the cube, hence the area of the surface is $6 \cdot 6^2 = 216\,\mathrm{m}^2$.

**Problem 2.** The two friends Christoph and Jonas hit the jackpot and bought a nice rectangular property of dimensions $35\,\mathrm{m}$ by $25\,\mathrm{m}$. They are planning to build a twin house and share garden $G$ of size $300\,\mathrm{m}^2$. The building floor plan can be seen in the picture:




(The distance between two neighboring grid lines is $5\,\mathrm{m}$.) How far must wall $b$ intrude from one section of the twin house into the other one so that the base areas of the friends' parts are equal?

*Result.* $8.75\,\mathrm{m}$

*Solution.* The base area of one house is half of $35\,\mathrm{m} \cdot 25\,\mathrm{m} - 300\,\mathrm{m}^2 = 575\,\mathrm{m}^2$, that is $287.5\,\mathrm{m}^2$. Since one side length of the rectangular house is $10\,\mathrm{m}$, the other side length must be $28.75\,\mathrm{m}$. Therefore, we get $b = 8.75\,\mathrm{m}$.

**Problem 3.** Little Marcus wants to go to the beach. He owns the following distinguishable beach-outfits: 5 swimming trunks, 3 straw hats, 4 sunglasses and 5 T-shirts. To comply with the beach rules, he has to wear swimming trunks. Wearing sunglasses, hats and T-shirts is not obligatory at all, but if he puts on some outfit, he always takes at most one of each category. How many different ways are there for Marcus to appear in an appropriate outfit?

*Result.* 600

*Solution.* Observe that the option of wearing nothing can be viewed as an additional outfit. Considering the hats, Marcus has to choose between not wearing a hat at all, wearing the first hat, the second one, or the third one, which gives 4 possibilities in total for the hats. Similarly, there are 5 possibilities to wear the sunglasses and 6 possibilities of wearing a T-shirt. Since Marcus has to put on one of the 5 swimming trunks, in total he has $5 \cdot 4 \cdot 5 \cdot 6 = 600$ possibilities to appear in an appropriate outfit.

**Problem 4.** Laura spent her vacation in a rain forest. Each day it either rained in the morning, or it rained in the afternoon, or it rained the whole day. Laura enjoyed altogether 13 days, when it did not rain all the time, but experienced exactly 11 morning rains and 12 afternoon rains. How long was Laura's vacation?

*Result.* 18 days

*Solution.* Let $v$ be the number of days of Laura's vacation. Then $v - 11$ is the number of days when it did not rain in the morning, and similarly $v - 12$ is the number of days when it did not rain in the afternoon. Since there was no day without any rain, we see that

$$(v - 11) + (v - 12) = 13$$

or $v = 18$.

**Problem 5.** Find the smallest non-negative integer solution of the equation $n - 2 \cdot \mathrm{Q}(n) = 2016$, where $\mathrm{Q}(n)$ is the sum of the digits of $n$.

*Result.* 2034

*Solution.* The number $n - \mathrm{Q}(n)$ is always divisible by 9. Since 2016 is divisible by 9, also $\mathrm{Q}(n)$ and consequently $n$ have to be divisible by 9. Clearly $n < 3000$, so $Q(n) \leq 2 + 9 + 9 + 9$, thus $n = 2016 + 2\,\mathrm{Q}(n) \leq 2074$. Now the only solution 2034 can easily be found.

**Problem 6.** How many positive integers have the property that their first (i.e. leftmost) digit is equal to their number of digits?

*Result.* 111 111 111

*Solution.* If $n$ is a non-zero digit, then there are exactly $10^{n-1}$ numbers starting with $n$ and fulfilling the property from the statement, for these are precisely the integers between $\overline{n0\ldots0}$ and $\overline{n9\ldots9}$. We conclude that there are

$$1 + 10 + \cdots + 100\,000\,000 = 111\,111\,111$$

such numbers in total.

**Problem 7.** A paving consists of many pavers, one of which has the shape of a regular $n$-gon, completely surrounded by other pavers. When this paver is rotated by $48°$ about its center, it fits again in its former position. What is the minimal $n$ for which this is possible?

*Result.* 15

*Solution.* A regular $n$-gon is preserved by a rotation precisely if it is by a multiple of the angle between the segments connecting two consecutive vertices with the center. This angle is $360°/n$, so we seek the smallest positive $n$ such that

$$\frac{48}{\frac{360}{n}} = \frac{2}{15}n$$

is an integer. The result is $n = 15$.

**Problem 8.** A day is called *happy* if its date written in the format *DD.MM.YYYY* consists of eight distinct digits—here *DD* fills in for the day, *MM* for the month and *YYYY* for the year, and if the day and the month is less than 10, a leading zero is prepended. For example, 26.04.1785 was a happy day. When is the next happy day (from now) going to be?

*Result.* 17.06.2345

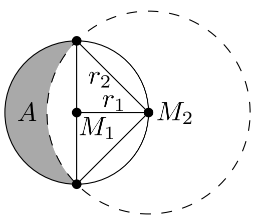
*Solution.* The month of a happy day either contains a zero or is 12, so either the year does not contain a zero or exceeds 3000. Let us pursue the former case. Since the leading digit of the day is one of 0, 1, 2, 3, we see that the year has to be at least 2145. However, this implies that the day is 30, which collides with the month. The second smallest possible year is 2345. We shall show that there is a happy day in that year. The leading digit of the day has to be 1, so the first possible month is 06. Finally, setting the day to be 17 is enough to complete the date of the happy day.

**Problem 9.** How many different planes contain exactly four vertices of a given cuboid?

*Result.* 12

*Solution.* There are six planes containing the faces of the cuboid, and further for each pair of opposite faces, there are two planes perpendicular to these faces and containing a face diagonal. In total there are 12 planes.

**Problem 10.** Little Sandra wants to draw a beautiful crescent using ruler and compass. First of all, she draws a circle with center $M_1$ and radius $r_1 = 3$ cm. Then she sets the compass at a point $M_2$ of this circle and draws a second circle with radius $r_2$ which meets the first circle in antipodal points of a diameter through $M_1$, as shown in the picture below.




What is the area of the crescent $A$ in $\text{cm}^2$?

*Result.* 9

*Solution.* To get the area of $A$, we have to subtract the area of the circular segment (center $M_2$, radius $r_2$) from the area of the semicircle (center $M_1$, radius $r_1$). For the area of the segment we calculate the area of the quadrant with radius $r_2$ and subtract the area of the isosceles right-angled triangle with leg $r_2$. Using the fact that $r_2^2 = 2r_1^2$ (the Pythagorean theorem), we infer that the sought area is

$$\frac{\pi r_1^2}{2} - \left(\frac{\pi r_2^2}{4} - r_1^2\right) = r_1^2 = 9\,\text{cm}^2.$$

**Problem 11.** All servants of King Octopus have six, seven, or eight legs. The ones having seven legs always lie, whereas the ones having six or eight legs always tell the truth. One day, King Octopus assembled four of his servants and asked them how many legs the four of them had altogether. The first servant reported that the total number of legs was 25, the next one claimed 26, the third one said 27 and the last one 28. How many legs do the the king's truth-telling servants (among these four) have in total?

*Result.* 6

*Solution.* Only one of the answers may be correct, so there are either three or four liars among the servants. However, if there were four of them, they would have 28 legs in total, implying that the last servant did not lie–a contradiction. So, the lying servants have 21 legs altogether. If the sole truth-telling servant had eight legs, the total number would be 29, which is not among the answers. We deduce that the truth-teller had six legs (and it was the third one to report the number of legs).

**Problem 12.** A shop sells bars of milk, white, and dark chocolate for the same price. One day, the shop earned 270 for the sold milk chocolate, 189 for the white chocolate, and 216 for the dark chocolate. What is the smallest total number of chocolate bars the shop could have sold on that day?

*Result.* 25

*Solution.* The price of a single chocolate bar is a common divisor of the amounts in the statement. Should the number of bars be minimal, the price has to be the greatest possible, i.e. the greatest common divisor. Since $\mathrm{GCD}(270, 189, 216) = 27$, we infer that the total number of sold bars is

$$\frac{270}{27} + \frac{189}{27} + \frac{216}{27} = 25.$$

**Problem 13.** A father of five children wants to have pastries for his family for tea time. Based on painful experience he knows that he has to distribute either the same type or five different types of pastry to his children, or else all kinds of heavy dispute will arise among the kids. One day, after a long discussion without any consent on the type of pastries, he exasperatedly instructed his youngest daughter Anna: "You'll go to the pastry shop and ask the salesgirl to give you $x$ pieces of pastries randomly! After you return home, each of the children shall get exactly one piece of pastry and the remaining pieces will be for mom and dad!" Assuming that the shop sells more than five types of pastry and it is always well stocked with every type, what number $x$ did the father choose in order to keep the peace among his children in any case and to keep the costs as low as possible at the same time?

*Result.* 17

*Solution.* If Anna ordered 16 or less pieces of pastry to be taken randomly, trouble among the kids could arise: For example in case of receiving 4 brownies, 4 blueberry muffins, 4 honey scones, and 4 danish, or less pieces of any of these, there are neither five pieces of the same type nor five pieces different from each other. On the other hand, if she asks for 17 random pieces, there might be five or more different types of pastry and the children would be happy. Otherwise there are at most four different types; however, in such a case, if there were less than five pieces of each type, there would be at most $4 \cdot 4 = 16$ pieces in total—a contradiction. We conclude that the father suggested Anna to ask the salesgirl for 17 randomly chosen pieces of pastry.

**Problem 14.** What is the ratio of the area of a circle to the area of a square, perimeters of which are equal?

*Result.* $4 : \pi$

*Solution.* Let $r$ be the radius of the circle and $a$ be the side length of the square. Since $2\pi r = 4a$, we compute the ratio of areas as

$$\frac{\pi r^2}{a^2} = \frac{2r \cdot \pi r}{a \cdot 2a} = \frac{2r \cdot 2a}{a \cdot \pi r} = \frac{4}{\pi}.$$

**Problem 15.** In February, Paul decided to visit the Cocos Islands with his private jet. He took off from his mansion in Europe at 10:00 Central European Time (CET) and landed on the Islands the next day at 5:30 local time (Cocos Islands Time, CCT). When returning home, he started at 8:30 CCT and landed at 17:00 CET the same day. Assuming that the duration of the flight was the same in both cases, what was the time on the Cocos Islands when Paul returned home?

*Result.* 22:30

*Solution.* Let $d$ be the duration of the flight and $s$ the difference between the time in Europe and on the Cocos Islands (in hours). The statement may be rewritten as the system of equations

$$d + s = 19.5,$$
$$d - s = 8.5$$

with the solution $d = 14$, $s = 5.5$. We deduce that Paul returned home at 22:30 of the Cocos Islands Time.

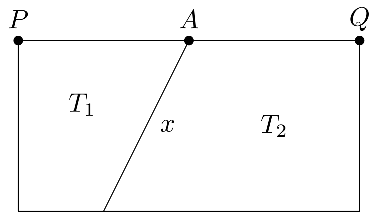
Note: The Cocos Islands indeed use the time zone GMT+6:30.

**Problem 16.** The numbers 14, 20, and $n$ fulfill the following condition: Whenever we multiply any two of them, the result is divisible by the third one. Find all positive integers $n$ for which this property holds.

*Result.* 70, 140, 280

*Solution.* Since $n$ divides $14 \cdot 20 = 2^3 \cdot 5 \cdot 7$, only the primes 2, 5, and 7 may occur in the factorization of $n$, with 5 and 7 occurring at most once and 2 at most three times. Further, from $14 \mid 20n$ wee see that $n$ is a multiple of 7, and similarly, $20 \mid 14n$ implies $10 \mid n$, so $70 \mid n$. It remains to conclude that all of the possible numbers 70, 140, 280 fulfill the conditions from the statement.

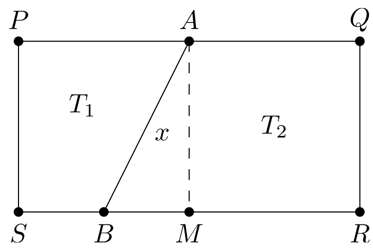
**Problem 17.** A rectangle is divided into two trapezoids along the line segment $x$ as in the picture below. The distance $PA$ is 10 cm and $AQ$ is 8 cm. The area of the trapezoid $T_1$ is $90\,\text{cm}^2$ and the area of $T_2$ is $180\,\text{cm}^2$.




What is the length of the segment $x$ in cm?

*Result.* 17

*Solution.* Denote by $R$, $S$ the other two vertices of the rectangle, $B$ the other endpoint of $x$, and $M$ the point on $SR$ such that $SM = PA = 10$.




As $PQ = 18$ and the area of rectangle $PQRS$ is $180 + 90 = 270$, it follows that $PS = QR = 270/18 = 15$. The formula for the area of trapezoid $T_2$ states that

$$180 = \frac{1}{2}(BR + AQ) \cdot QR$$

or $BR = 16$. Now $BM = BR - MR = 8$ and using the Pythagorean theorem,

$$x = \sqrt{AM^2 + BM^2} = \sqrt{289} = 17.$$

**Problem 18.** Elisabeth has harvested strawberries in her garden. She wants to distribute them to her four sons in such a way that each son gets at least three strawberries and Valentin receives more strawberries than Benedikt, Benedikt more than Ferdinand, and Ferdinand more than Michael. Each son knows his number of strawberries, the total number of strawberries distributed, and the above-mentioned conditions. How should Elisabeth distribute the strawberries in order to hand as few as possible of them and none of her sons is able to determine the whole distribution?

*Result.* $(M, F, B, V) = (3, 5, 6, 8)$

*Solution.* Let us denote by $V$ the number of Valentin's strawberries; obviously, $V \geq 6$. By analysing cases, we will show that Elisabeth cannot distribute less than 22 strawberries. If $V = 6$, there is only one possible distribution, namely $(3, 4, 5, 6)$. In the case $V = 7$, each of the possible distributions $(3, 4, 5, 7)$, $(3, 4, 6, 7)$, $(3, 5, 6, 7)$, and $(4, 5, 6, 7)$ uses a different number of strawberries, therefore Valentin, knowing the total number, can determine the distribution. Similarly, for $V = 8$ or $V = 9$, only the distributions $(3, 4, 5, 8)$, $(3, 4, 6, 8)$, and $(3, 4, 5, 9)$ exist (with less than 22 strawberries), each one being computable by Valentin.

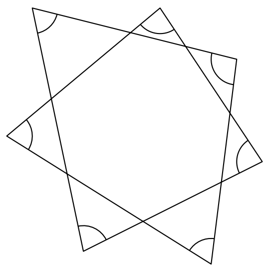
On the other hand, the distribution $(3, 5, 6, 8)$ satisfies all the conditions: Valentin and Michael cannot distinguish it from $(3, 4, 7, 8)$, whereas Ferdinand and Benedikt can think that the distribution is $(4, 5, 6, 7)$. It remains to show that no other distribution of 22 strawberries complies with the requirements: From $(3, 4, 5, 10)$, $(3, 4, 6, 9)$, $(3, 4, 7, 8)$, and $(4, 5, 6, 7)$, the third one can be deduced by Benedikt and the remaining three by Valentin.

**Problem 19.** We write all the integers from 1 to 1000 consecutively clockwise along the circumference of a circle. We now mark some of the numbers: Starting with 1, go clockwise and mark every 15th number (i.e. 16, 31 etc.). We continue this way, until we are forced to mark a number which we have already marked. How many numbers stay unmarked at the end of the procedure?

*Result.* 800

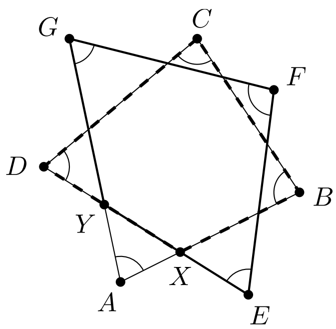
*Solution.* In the first pass, all the numbers of the form $15k+1$ (for some integer $k$) are marked, starting with 1 and ending with 991; the following pass starts with 6 and ends with 996, marking all the numbers of the form $15k+6$. Finally, in the third pass one begins with the number 11 and ends with 986 (the marked numbers having the form $15k+11$), which is the last number to be marked. Observe that we have marked exactly all the numbers of the form $5k+1$, which comprise precisely one fifth of all the numbers on the circle. We conclude that $4/5 \cdot 1000 = 800$ numbers remain unmarked.

**Problem 20.** Find the sum of the seven marked interior angles of this 7-pointed star (in degrees)!



*Result.* 540°

*Solution.* Denote the tips of the star by $A, B, \dots, G$ as in the picture; further, let $X$, $Y$ be the intersection of $DE$ with $AB$, $AG$, respectively.




Let $S$ be the sum in question. Since the sum of the internal angles in both quadrilaterals $XBCD$ and $YEFG$ is 360°, we see that

$$S + \angle BXY + \angle XYG - \angle XAY = 2 \cdot 360^\circ.$$

However, $\angle BXY = 180^\circ - \angle AXY$ and $\angle XYG = 180^\circ - \angle XYA$, so

$$\angle BXY + \angle XYG - \angle XAY = 360^\circ - (\angle AXY + \angle XYA + \angle XAY) = 180^\circ.$$

It follows that $S = 540^\circ$.

**Problem 21.** Pupils were given the following exercise: They should compute the arithmetic mean of the numbers 1, 3, 6, 7, 8, and 10. However, Lucy chose a wrong approach: First she picked two of the numbers and computed their arithmetic mean. Then she computed the arithmetic mean of the result and some other number and repeated this step until she had used all the numbers. What is the largest absolute value of the error (i.e. the difference with the correct result) Lucy could have achieved?

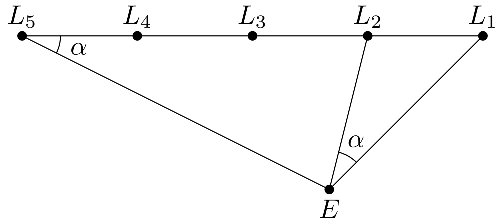
*Result.* 17/6

*Solution.* One can easily see that Lucy's procedure is in fact the following: She picks some ordering of the given numbers, say $(a_1, a_2, a_3, a_4, a_5, a_6)$, and computes

$$S = \frac{a_1}{2^5} + \frac{a_2}{2^5} + \frac{a_3}{2^4} + \frac{a_4}{2^3} + \frac{a_5}{2^2} + \frac{a_6}{2^1}.$$

Among all these orderings, the highest value of $S$ is achieved for the ascending ordering, since the largest number is divided by the smallest, the second largest by the second smallest etc. Similarly, the smallest value of $S$ is achieved for the descending ordering. Obviously, the largest error will occur for one of these extremal orderings. The arithmetic mean of the given numbers is 35/6. If the ordering is chosen ascending, we get $S = 67/8$, yielding the error 61/24. If the ordering is descending, we obtain $S = 3$ with the error 17/6, which is the greater of the two and thus the sought result.

**Problem 22.** Along one side of a straight road there are five street lights $L_1$, $L_2$, $L_3$, $L_4$, and $L_5$ lined up equally spaced 12 m apart. On the other side of the road there is an ice cream shop. If Julien is standing at the entrance $E$ of the shop, the angle subtended at this point by $L_1$ and $L_2$ is $\alpha = 27°$. If he is standing at $L_5$, the angle at that point subtended by $L_1$ and $E$ is $27°$, too.




What is the distance from $L_1$ to $E$?

*Result.* 24 m

*Solution.* The triangles $EL_1L_2$ and $EL_1L_5$ are similar, since $\alpha$ and $\angle L_5L_1E$ occur as interior angles in both of them. Therefore, we get

$$\frac{EL_1}{L_2L_1} = \frac{L_5L_1}{EL_1} \quad \text{or} \quad EL_1^2 = L_2L_1 \cdot L_5L_1 = 12 \cdot 48 = 576,$$

yielding the desired distance $EL_1 = 24$ m.

**Problem 23.** Clara chose two distinct integers from 1 to 17, inclusive, and multiplied them. Surprisingly, the product turned out to be equal to the sum of the remaining fifteen numbers. Find Clara's two numbers.

*Result.* 10 and 13

*Solution.* Denote by $a$ and $b$ the numbers fulfilling the given condition. As the sum of the first 17 numbers is 153, we have to solve the equation $153 - (a + b) = ab$. Rearranging and adding 1 on both sides of the equation leads to $154 = ab + a + b + 1 = (a + 1)(b + 1)$. Since $154 = 2 \cdot 7 \cdot 11$ and $2 \leq (a + 1), (b + 1) \leq 18$, the only possible factorization is $154 = 11 \cdot 14$. Therefore, the sought-after numbers are 10 and 13.

**Problem 24.** How many 6-tuples $(a, b, c, d, e, f)$ of positive integers satisfy $a > b > c > d > e > f$ and $a + f = b + e = c + d = 30$ simultaneously?

*Result.* $\binom{14}{3} = 364$

*Solution.* Let us express the 6-tuple as

$$(a, b, c, d, e, f) = (15 + x, 15 + y, 15 + z, 15 - z, 15 - y, 15 - x),$$

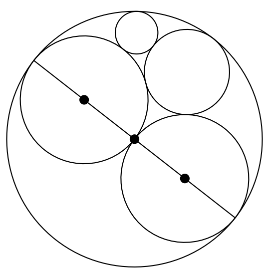
where $0 \leq x, y, z < 15$. The condition $a > b > c > d > e > f$ is equivalent to $x > y > z > 0$, so the 6-tuple is uniquely determined by a choice of three positive integers less than 15. It follows that there are $\binom{14}{3} = 364$ such 6-tuples.

**Problem 25.** A timed bomb is equipped with a display showing the time before the explosion in minutes and seconds. It starts counting down with the value 50:00 on the display. A light bulb blinks whenever the displayed number of remaining minutes is equal to the displayed number of remaining seconds (e.g. 15:15) or when the four digits on the display read the same when reversed (e.g. 15:51). We can disable the bomb when the light blinks for the 70th time. What will be the time on the display then?

*Result.* 03:03

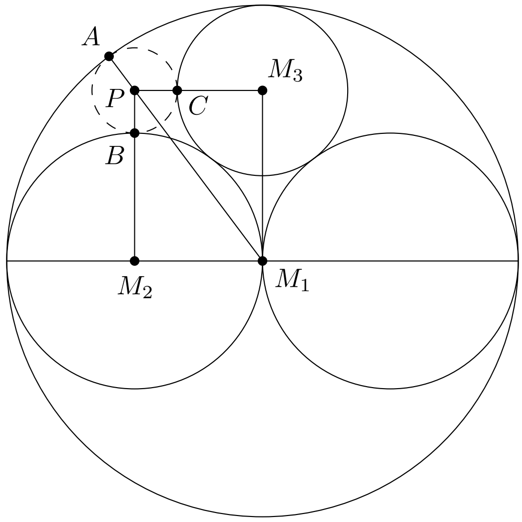
*Solution.* The number of minutes is equal to the number of seconds once in each minute, so in 50 minutes this happens 50 times. The event when the number on the display reads the same occurs once in each minute with the units digit not exceeding 5, so this happens 30 times. There are five cases when both these events happen at once: $00{:}00, 11{:}11, \ldots, 44{:}44$. Therefore the light bulb blinks $50 + 30 - 5 = 75$ times (including at 00:00) before the bomb explodes; we can disable the bomb when there are only five blinks remaining (00:00, 01:01, 01:10, 02:02, 02:20), i.e. when there is 03:03 shown on the display.

**Problem 26.** Five circles are tangent to each other as indicated in the figure. Find the radius of the smallest circle, if the radius of the big circle is 2 and the two other circles with marked centers are of radius 1.



*Result.* $\frac{1}{3}$

*Solution.* Denote by $M_1$, $M_2$, and $M_3$ the centers of the circles as in the figure and by $r_3$ the radius of the second smallest circle.




Due to the symmetry of the whole figure, $M_1M_2 \perp M_1M_3$ and the Pythagorean theorem helps to find $r_3 = \frac{2}{3}$ from the equation

$$M_1M_2^2 + M_1M_3^2 = M_2M_3^2 \quad \text{or} \quad 1 + (2 - r_3)^2 = (1 + r_3)^2.$$

Let $P$ be a point that completes the centers $M_1$, $M_2$, and $M_3$ to a rectangle. Let $A$, $B$, and $C$ be the intersection points of rays $M_1P$, $M_2P$, and $M_3P$ with the respective circles. Since $M_2M_1M_3P$ is a rectangle, we get $PB = \frac{4}{3} - 1 = \frac{1}{3}$, $PC = 1 - \frac{2}{3} = \frac{1}{3}$ and $PA = 2 - M_2M_3 = 2 - (1 + r_3) = \frac{1}{3}$. Therefore, $P$ has distance $\frac{1}{3}$ to all three points $A$, $B$, and $C$. Due to this fact, these points lie on the circle with center $P$ and radius $\frac{1}{3}$. Since $M_1P$, $M_2P$, and $M_3P$ are straight lines, the points $A$, $B$, and $C$ are the tangent points of the corresponding circles and the circle with center $P$ and radius $\frac{1}{3}$ is the small circle in the picture of the posed problem.

**Problem 27.** In a casino, some people sat around a large table playing roulette. When Erich left that table carrying his assets of 16 000 euros away, the average balance of all players decreased by 1 000 euros. It diminished again by 1 000 euros, when the two gamblers Bettina and Elfi got into business at that table joining in with 2 000 euros each. How many players sat around the table while Erich was still gambling?

*Result.* 9

*Solution.* By $n$ denote the number of people at the beginning, when Erich still was playing, and by $x$ denote the average balance of one gambler at that table. From the given statements we obtain the following two equations:

$$\frac{nx - 16\,000}{n-1} = x - 1\,000 \quad \text{and} \quad \frac{nx - 16\,000 + 2 \cdot 2\,000}{n+1} = x - 2 \cdot 1\,000.$$

Working out these equations yields

$$x = 17\,000 - 1\,000n \quad \text{and} \quad 2\,000n - 10\,000 = x,$$

and now we easily obtain $n = 9$. So, while Erich was gambling, there were nine people sitting at the table.

**Problem 28.** In a cube $7 \times 7 \times 7$, each two neighboring unit cubes are separated by a partition. We want to remove some of the partitions so that each unit cube will become connected with at least one of the outer unit cubes. What is the minimum number of partitions to be removed?

*Result.* 125

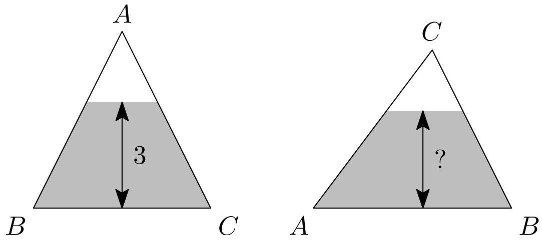
*Solution.* At the beginning, there are $7^3$ unit cubes. By removing one partition we connect two unit cubes and the number of isolated spaces within the cube decreases by one. At the end, we want to have at most $7^3 - 5^3$ isolated spaces (that is the number of outer unit cubes). It follows that we need to remove at least $5^3 = 125$ partitions. It is easy to see that 125 is enough.

**Problem 29.** It is known that 20***16 is a 7-digit square of an integer. What are the three missing digits?

*Result.* 909

*Solution.* Let $a^2$ be a perfect square ending with $\ldots 16$. This means that $a^2 - 16 = (a-4)(a+4)$ is divisible by 100, so $a = 2b$ and $(b-2)(b+2)$ is divisible by 25. Thus, $b = 25n \pm 2$ and consequently $a = 50n \pm 4$. As $1404^2 < (1.414 \cdot 1000)^2 < (1000\sqrt{2})^2 = 2000000$ and $1454^2 > 1450^2 = 2102500 > 2100000$, the only possibility is $a = 1446$, yielding $a^2 = 2090916$.

**Problem 30.** Triangle $ABC$ with $AB = AC = 5\,\mathrm{m}$ and $BC = 6\,\mathrm{m}$ is partially filled with water. When the triangle lies on the side $BC$, the surface of the water is $3\,\mathrm{m}$ above the side. What is the height in meters of the area filled with water when the triangle lies on the side $AB$?




*Result.* 18/5

*Solution.* Let $D$ be the midpoint of $BC$; then $\triangle ABD$ is a right-angled triangle, so from the Pythagorean theorem, $AD = 4$. The part of the triangle not filled with water is therefore a triangle similar to $\triangle ABC$ with ratio of similitude $1/4$. Since the ratio of the areas (of the non-filled part and the whole triangle) stays the same after rotating the triangle, an analogous similarity has to occur in the new situation, too. Therefore the surface of the water is always in $3/4$ of the height, so it suffices to compute the height from $AB$. Knowing that the area of $\triangle ABC$ is $\frac{1}{2} \cdot AD \cdot BC = 12$, we have $h_{AB} = 2 \cdot 12/AB = 24/5$. We conclude that the area filled with water is of height $3/4 \cdot 24/5 = 18/5$.

**Problem 31.** There are six boxes numbered 1 to 6 and 17 peaches somehow distributed in them. The only move we are allowed to do is the following: If there are exactly $n$ peaches in the $n$-th box, we eat one of them and add the remaining $n-1$ peaches to the boxes 1 to $n-1$, one to each box. What is the distribution of the peaches provided that we can eat all the peaches?

*Result.* $1, 1, 3, 2, 4, 6$

*Solution.* Let us trace backwards the possible moves: The final state $(0,0,0,0,0,0)$, i.e. when all the peaches are eaten, can be reached only from $(1,0,0,0,0,0)$, which in turn could have emerged only from $(0,2,0,0,0,0)$ etc.—this way we construct a unique chain of distributions of peaches

$$\ldots (0,2,0,0,0,0),\ (1,2,0,0,0,0),\ (0,1,3,0,0,0),\ (1,1,3,0,0,0),\ \ldots$$

ending with $(1,1,3,2,4,6)$, which is the sought-after distribution of seventeen peaches.

**Problem 32.** A simple aerial lift with fixed two-person chairs operates on a mountain. 74 people are planning to travel upwards, whereas 26 passengers are waiting at the upper station. Exactly at noon, the lift starts working and a pair of people gets on the lift on both its stations; the rest of the passengers is then loaded continuously. At 12:16, the leading chair going upwards meets the last occupied chair going downwards, and at 12:22, the leading chair going downwards meets the last occupied chair going upwards. The distance between each two chairs on the rope is the same, the lift maintains constant speed, and all the passengers travel in pairs. How long does it take from the lower station to the upper (in minutes)?

*Result.* 26

*Solution.* Firstly, the distance between the first and the last up-going chair is thrice the distance of the first and the last down-going chair. Thus, we see that the time between the two moments described in the statement is twice the time from the moment the leading chairs meet till the moment the leading up-going chair meets the last down-going one. We infer that the leading chairs met at 12:13 (exactly in the middle of the lift), hence the time needed to go through the whole lift is 26 minutes.

**Problem 33.** Let $ABCD$ be a rhombus and $M$, $N$ points on the segments $AB$, $BC$ different from $A$, $B$, $C$ such that $DMN$ is an equilateral triangle and $AD = MD$. Find $\angle ABC$ (in degrees).

*Result.* $100°$

*Solution.* Since $CD = AD = MD = ND$, the triangles $AMD$ and $NCD$ are isosceles with bases $AM$, $NC$, respectively. Put $\theta = \angle DAB$; then $\angle ABC = \angle ADC = 180° - \theta$. On the other hand, since

$$\angle DAM = \angle AMD = \angle DNC = \angle NCD = \theta,$$

we have

$$\angle ADM = \angle NDC = 180° - 2\theta$$

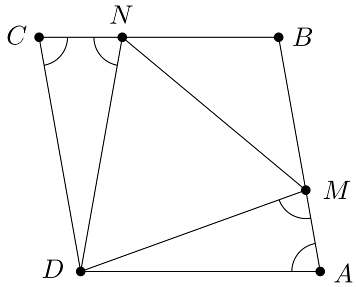
and

$$\angle ADC = \angle ADM + \angle MDN + \angle NDC = 420° - 4\theta.$$

We conclude that

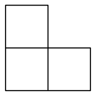
$$420° - 4\theta = 180° - \theta$$

or $\theta = 80°$, therefore $\angle ABC = 100°$.




**Problem 34.** In how many ways is it possible to color the cells of a $2 \times 7$ table with green and yellow in such a way that neither green nor yellow L-trimino appears in the table?

Note: L-trimino is the following (possibly rotated) shape:



*Result.* 130

*Solution.* If any column of the table is monochromatic, then the neighboring column(s) must have the other color, so the next column(s) must have the same color as the first one etc., so there are two possibilities to color the table this way, according to which color is used in the first column.

On the other hand, the previous paragraph implies that if there is a column colored with both colors, all the other columns have to use both colors as well. It is easy to see that no matter how we distribute the colors between the upper and lower cells in this case, the resulting coloring will always satisfy the condition from the statement, so there are $2^7 = 128$ such colorings.

In total there are $2 + 128 = 130$ colorings of the table.

**Problem 35.** Michael is a keen diamond collector, but so far he owns less than 200 diamonds. He divided all his diamonds into several (at least two) piles in such a way that

- each two piles consist of different number of diamonds,
- none of these piles consists of exactly two diamonds,
- for each of these piles it holds that whenever it is divided into two smaller piles, at least one of these new piles has the same size as some previously existing one.

What is the greatest number of diamonds Michael can possess?

Note: A pile consists of a non-zero number of diamonds.

*Result.* 196

*Solution.* Assume that we have a division in accordance with the problem statement. Let $m$ be the number of diamonds in the smallest pile. If $m \geq 2$, then the smallest pile can be divided into two piles of sizes 1 and $m - 1$ respectively, neither of which has the same size as some other pile; hence $m = 1$.

Next, we show that the second smallest pile consists of 3 diamonds. Since 2 is excluded, we have to rule out the case when it consists of $n \geq 4$ diamonds. However, this is clearly not possible because of the division $n = 2 + (n - 2)$.

Finally, we prove that if the piles $1, 3, \ldots, 2k - 1$ ($k > 1$) are the $k$ smallest piles in the division, then the $(k + 1)$-th smallest pile (if it exists) consists of $2k + 1$ diamonds. Let $p$ be the size of the $(k + 1)$-th smallest pile. Clearly, $p$ is odd, for a pile of even size could be divided into two smaller ones of even size. If $p \geq 2k + 3$, the division $p = 2 + (p - 2)$ yields a contradiction. We conclude that the only possibility is $p = 2k + 1$, which is readily seen to satisfy the conditions from the statement.

It follows by induction that the number of Michael's diamonds is of the form $1 + 3 + \cdots + (2k - 1) = k^2$. The largest square number less than 200 is $14^2 = 196$, which is the largest possible number of diamonds in Michael's possession.

**Problem 36.** Recall that in the game of rock-paper-scissors we have three shapes: $R$ – rock, $P$ – paper and $S$ – scissors such that $S > P$, $P > R$, $R > S$ and $R = R$, $P = P$, $S = S$, where $A > B$ means '$A$ beats $B$' and $A = B$ means 'when $A$ is played against $B$, the game ends in a tie'. A tournament in Two Handed Rock-Paper-Scissors Without Repetition between players $P_1$ and $P_2$ consists of 9 games. In every game each player chooses a pair $(\ell_i, r_i)$ where $\ell_i$ and $r_i$ stand for shapes played by the left and right hand, respectively, of the player $P_i$. During the whole tournament each player must choose every possible pair exactly once. In a single game we distribute 4 points in the following way: the winner on each pair of playing hands (left/right) receives 2 points and the loser receives 0 points or both player receive 1 point if there is a tie in a pair of hands. Suppose that players are choosing their moves at random. What is the probability that each of the 9 games in the tournament ends in a tie (i.e. with score 2:2)?

*Result.* $3!^3/9! = 1/1680$

*Solution.* Let us define three sets of three pairs each:

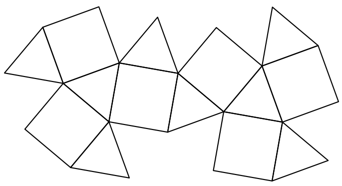
$$D_R = \{(R,R),(P,S),(S,P)\}, \quad D_P = \{(P,P),(S,R),(R,S)\}, \quad D_S = \{(S,S),(R,P),(P,R)\}.$$

Note that a single game in the tournament ends in a tie if and only if two pairs from the same set out of $D_R$, $D_P$, $D_S$ are played against each other.

Possible outcomes of the whole tournament are all pairs of permutations of the set $D_R \cup D_P \cup D_S$. All the games end in a tie if and only if elements of each set $D_R$, $D_P$, $D_S$ occupy the same three positions in $P_1$'s and $P_2$'s permutation. Consider any permutation and let it denote the order of moves of the player $P_1$ in subsequent games. Number of arrangements of $P_2$'s moves yielding a draw in each game equals $3!^3$, no matter what the $P_1$'s permutation was. Therefore the desired probability is

$$\frac{3!^3}{9!} = \frac{1}{1680}.$$

**Problem 37.** The net of a solid consists of eight regular triangles and six squares, as shown in the picture:



Assuming that the length of each edge is 1 km, what is the volume of the solid (in km$^3$)?

*Result.* $\frac{5}{3}\sqrt{2}$

*Solution.* The described solid can be obtained from a cube in the following way: Each corner of the cube is cut off, the cut going through the centers of the edges adjacent to the removed vertex. The edge length of the cube is $\sqrt{2}$, so its volume is $2\sqrt{2}$. The removed corners are eight (oblique) pyramids, the base of each being an isosceles right-angled triangle with the leg of length $\sqrt{2}/2$, and the height being $\sqrt{2}/2$, too. Therefore the volume of one corner is $\frac{1}{3} \cdot \frac{1}{2} \cdot (\sqrt{2}/2)^2 \cdot (\sqrt{2}/2) = \sqrt{2}/24$ and the volume of the given solid is $2\sqrt{2} - 8 \cdot \sqrt{2}/24 = 5\sqrt{2}/3$.

**Problem 38.** Find the only three-digit prime factor of 999 999 995 904.

*Result.* 601

*Solution.* Observe that

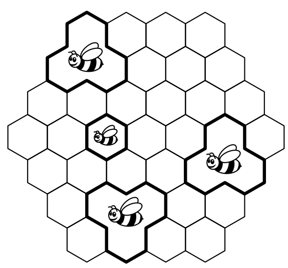
$$999\,999\,995\,904 = 10^{12} - 2^{12} = 2^{12}(5^{12} - 1)$$

and

$$5^{12} - 1 = (5-1)(5+1)(5^2+1)(5^2-5+1)(5^2+5+1)(5^4-5^2+1),$$

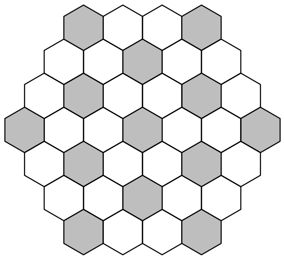
but only the last factor is greater than 100. Since we know that a three-digit prime factor exists and $5^4 - 5^2 + 1 = 601$ is clearly divisible by neither of 2, 3, 5, it is a prime, and hence the sought number.

**Problem 39.** Thirteen bees: one little bee and twelve large bees are living on a 37-cell honeycomb. Each large bee occupies 3 pairwise adjacent cells and the little bee occupies exactly 1 cell (see the picture). In how many ways can the honeycomb be divided into 13 non-overlapping sectors so that all thirteen bees can be accommodated in accordance with the given restrictions?



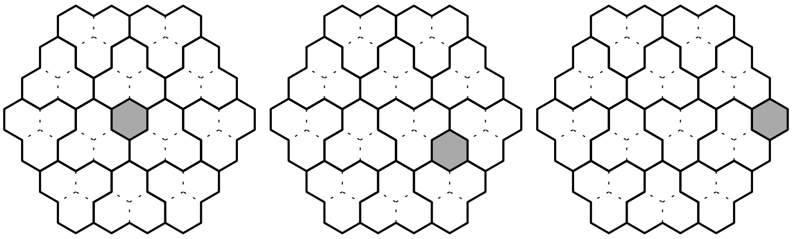
*Result.* 20

*Solution.* Let us consider 13 cells shaded in the picture below:



Each three-cell sector contains exactly one shaded cell, so the cell of the little bee must be one of the shaded.

If it is the central cell, then there are exactly two ways to divide the rest of the honeycomb into 12 large bee sectors (the one shown in the picture and the one rotated by 60 degrees). For each of the 6 'middle' shaded cells there is exactly one way to accommodate large bees in the rest of the honeycomb. Finally, for each of the boundary shaded cells there are exactly two ways to divide the remaining cells into three-cells sectors (the one shown and the symmetric one).

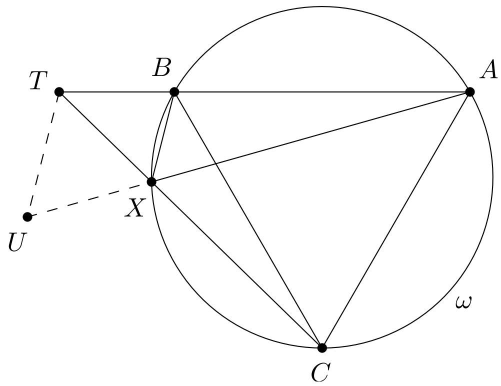


This gives a total of $2 + 6 \cdot 1 + 6 \cdot 2 = 20$ ways to dissect the honeycomb into the sectors as requested.

**Problem 40.** Equilateral triangle $ABC$ is inscribed in circle $\omega$. Point $X$ is on the (shorter) arc $BC$ of $\omega$ and $T$ is the intersection of $AB$ and $CX$. If $AX = 5$ and $TX = 3$, find $BX$.

*Result.* 15/8

*Solution.* Since $\angle AXB = \angle ACB = 60^\circ$ and $\angle AXC = \angle ABC = 60^\circ$, $\angle BXT = 180^\circ - \angle AXB - \angle AXC = 60^\circ$. Let $U$ be a point on $AX$ such that $TU \parallel BX$.




Then $TUX$ is an equilateral triangle and $\triangle TUA \sim \triangle BXA$. Therefore we have

$$BX = \frac{TU}{AU} \cdot AX = \frac{TX \cdot AX}{TX + AX} = \frac{15}{8}.$$

**Problem 41.** Let $ABC$ be an equilateral triangle. An interior point $P$ of $ABC$ is said to be *shining* if we can find exactly 27 rays emanating from $P$ intersecting the sides of the triangle $ABC$ such that the triangle is divided by these rays into 27 smaller triangles of equal area. Determine the number of shining points in $ABC$.

*Result.* $\binom{26}{2} = 325$

*Solution.* We see that $PA$, $PB$, $PC$ are among the 27 rays from $P$: If not, we would obtain a quadrilateral, leading to a contradiction. Let us divide the perimeter of $\triangle ABC$ into 27 segments such that each side is divided into segments of equal length; there are altogether $\binom{26}{2} = 325$ such divisions, since if we fix $A$ to be the first dividing point, we can freely choose $B$ and $C$ from the remaining 26 points. Finally, observe that each such division corresponds to exactly one shining point and vice versa: Clearly, (the rays from) each shining point gives rise to a division. On the other hand, given the numbers $a$, $b$, $c$ of the segments the respective sides are divided into, we let $P$ be the unique point inside $\triangle ABC$ such that the distances of $P$ to the sides $BC$, $CA$, $AB$ are in the ratio $a : b : c$. A straightforward computation shows that $P$ is indeed the shining point, the rays from which divide $\triangle ABC$ in accordance with the given division.

**Problem 42.** How many positive divisors of $2016^2$ less than 2016 are not divisors of 2016?

*Result.* 47

*Solution.* From the prime factorization $2016 = 2^5 \cdot 3^2 \cdot 7$ we obtain $2016^2 = 2^{10} \cdot 3^4 \cdot 7^2$. Therefore, $2016^2$ has $11 \cdot 5 \cdot 3 = 165$ positive divisors of which $\frac{1}{2} \cdot (165 - 1) = 82$ are less than 2016—excluding 2016, the divisors may be divided into pairs $(x, y)$ such that $x \cdot y = 2016^2$ and $x < 2016 < y$. Note that 2016 has $6 \cdot 3 \cdot 2 - 1 = 35$ divisors less than 2016 which naturally are divisors of $2016^2$, too. Hence, the desired number of divisors is $82 - 35 = 47$.

**Problem 43.** Let

$$Z_n = \frac{4n + \sqrt{4n^2 - 1}}{\sqrt{2n - 1} + \sqrt{2n + 1}}.$$

Compute $Z_1 + Z_2 + \cdots + Z_{2016}$.

*Result.* $\frac{1}{2}(4033\sqrt{4033} - 1)$

*Solution.* Observe that for $n \in \mathbb{N}$,

$$\begin{aligned}
\frac{4n + \sqrt{4n^2 - 1}}{\sqrt{2n - 1} + \sqrt{2n + 1}} &= \frac{(\sqrt{2n+1} - \sqrt{2n-1})((\sqrt{2n+1})^2 + (\sqrt{2n+1})(\sqrt{2n-1}) + (\sqrt{2n-1})^2)}{(\sqrt{2n+1} - \sqrt{2n-1})(\sqrt{2n+1} + \sqrt{2n-1})} \\
&= \frac{1}{2}((\sqrt{2n+1})^3 - (\sqrt{2n-1})^3).
\end{aligned}$$

Therefore,

$$\begin{aligned}
Z_1 + \cdots + Z_{2016} &= \frac{1}{2}\big((\sqrt{3})^3 - (\sqrt{1})^3 + (\sqrt{5})^3 - (\sqrt{3})^3 + \cdots + (\sqrt{4033})^3 - (\sqrt{4031})^3\big) \\
&= \frac{1}{2}(4033\sqrt{4033} - 1).
\end{aligned}$$

**Problem 44.** We construct a sequence of integers $a_0, a_1, a_2 \ldots$ in the following way: If $a_i$ is divisible by three, let $a_{i+1} = a_i/3$; otherwise let $a_{i+1} = a_i + 1$. For how many different positive integers $a_0$ does the sequence reach the value 1 for the first time in exactly eleven steps (i.e. $a_{11} = 1$, but $a_0, a_1, \ldots, a_{10} \neq 1$)?

*Result.* 423

*Solution.* The number 1 can be reached only from 3, which in turn can be obtained from 2 or 9. Further, 9 can arise from 8 or 27; 2 can be reached from 1 or 6, but only 6 is admissible in the light of the problem statement. Let us construct further predecessors: Let $P_n$ be the set of positive integers such that the sequence from the statement reaches 1 in exactly $n$ steps if and only if $a_0 \in P_n$. Clearly, to establish $P_{n+1}$ for $n \geq 3$, we take $3x$ for each $x \in P_n$ and also $x - 1$ for each $x \in P_n$ such that $x - 1$ is not a multiple of three.

Let $p_n$ be the number of elements of $P_n$, and denote further by $f_n$, $g_n$, $h_n$ the number of elements of $P_n$ of the form $3k$, $3k + 1$, $3k + 2$, respectively. Observe that for $n \geq 3$, all the elements of $P_n$ are greater than 3, and so

- $f_{n+1} = p_n$, since for each $x \in P_n$ there is $3x \in P_{n+1}$,
- $g_{n+1} = h_n$, since for each $x \in P_n$ of the form $3k + 2$ there is $x - 1 = 3k + 1 \in P_{n+1}$, and
- $h_{n+1} = f_n$ for similar reasons.

Therefore we have

$$p_n = f_n + g_n + h_n = p_{n-1} + p_{n-2} + p_{n-3}$$

for $n \geq 4$. The initial calculations show that $p_1 = 1$, $p_2 = 2$, and $p_3 = 3$, and the subsequent terms can be calculated in a straightforward manner using the recurrence above. The sought result is $p_{11} = 423$.

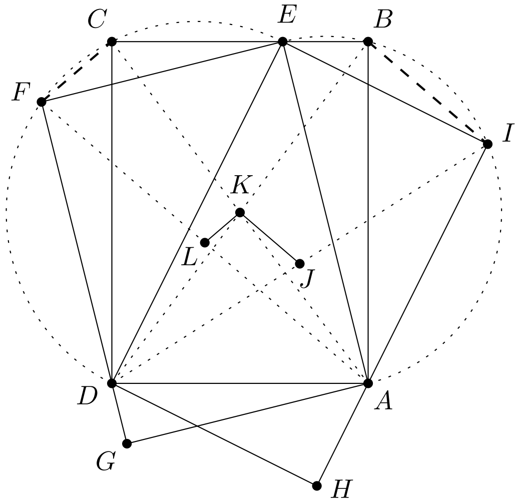
**Problem 45.** Let $ABCD$, $AEFG$, and $EDHI$ be rectangles with centers $K$, $L$, $J$, respectively. Assume further that $A$, $D$, $E$ are inner points of line segments $HI$, $FG$, $BC$, respectively, and $\angle AED = 53°$. Determine the size of $\angle JKL$ (in degrees).

*Result.* 74°

*Solution.* Since $KJ$ is a median in triangle $BID$, we have $KJ \parallel BI$, and similarly, $KL \parallel CF$. Thus $\angle JKL = \angle IBA + \angle DCF$. Since $\angle AIE = \angle ABE = 90°$, the quadrilateral $BIAE$ is cyclic. Consequently,

$$\angle IBA = \angle IEA = 90° - \angle AED = 37°.$$

In the same way we deduce $\angle DCF = 37°$, therefore $\angle JKL = 74°$.




**Problem 46.** James has picked several (not necessarily distinct) integers from the set $\{-1, 0, 1, 2\}$ in such a way that their sum equals 19 and the sum of their squares is 99. What is the greatest possible value of the sum of the cubes of James' numbers?

*Result.* 133

*Solution.* Assume that there are exactly $a$, $b$, $c$ numbers equal to $-1$, 1, 2, respectively, among the James' numbers (those equal to 0 clearly play no role). The conditions from the statement may then be rewritten as

$$-a + b + 2c = 19,$$
$$a + b + 4c = 99.$$

Our goal is to maximize $-a + b + 8c = 19 + 6c$. However, by adding the equalities, we find out that $6c = 118 - 2b$, so $c \le 19$. The value $c = 19$ can be obtained with the choice $a = 21$, $b = 2$, so the sought maximum is $19 + 6 \cdot 19 = 133$.

**Problem 47.** Find the largest 9-digit number with the following properties:

- all of its digits are different;
- for each $k = 1, 2, \ldots, 9$, when the $k$-th digit is crossed out the resulting 8-digit number is divisible by $k$.

*Result.* 876 513 240

*Solution.* Denote by $A_k$ the $k$-th digit of the sought number, so the number is $\overline{A_1A_2A_3A_4A_5A_6A_7A_8A_9}$. There are 10 possible digits, so exactly one, say $d$, is not used in the decimal representation of the desired number. Let $N_k$ be the number with $k$-th digit crossed out.

We know that $N_2$ is even, so $2 \mid A_9$. Furthermore, the number $N_5$ is divisible by 5, hence so is $A_9$. This means that $A_9 = 0$.

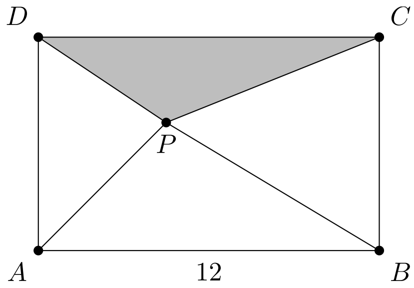
The number $N_9$ is divisible by 9, so the digit sum of this number, namely $1+2+3+4+5+6+7+8+9-d = 45-d$, is divisible by 9, which leads to $d = 9$.

Numbers $N_8$ and $N_4$ are both divisible by 4 which means that both digits $A_7$, $A_8$ are even. In addition, the first of these numbers is divisible by 8, so the 2-digit number $\overline{A_6A_7}$ is divisible by 4. Also numbers $N_3$ and $N_6$ are divisible by 3 and so is the digit sum of the desired number. From this we get $\{A_3, A_6\} = \{3, 6\}$.

As we are looking for the largest possible number, suppose that $A_1 = 8$, $A_3 = 6$, $A_6 = 3$. We then have $\{A_7, A_8\} = \{2, 4\}$ but since $4 \mid \overline{A_6A_7}$, $A_7 = 2$ and $A_8 = 4$.

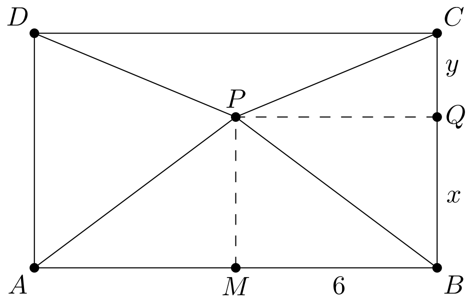
It suffices to check that putting the remaining digits in a decreasing order in the gaps leads to the number 876513240 which satisfies the remaining condition, i.e. the number $N_7 = 87651340$ is divisible by 7.

**Problem 48.** Point $P$ lies inside a rectangle $ABCD$ with $AB = 12$. Each of triangles $ABP$, $BCP$, $DAP$ has its perimeter equal to its area. What is the perimeter of triangle $CDP$?




*Result.* 25

*Solution.* Note that a triangle has equal area and perimeter if and only if the incircle has radius 2. Thus triangles $BCP$ and $ADP$ are congruent; indeed, if $P$ is closer to $AD$ than $BC$, then the inradius of $ADP$ is smaller than the inradius of $BCP$. This means that $P$ lies on one of the axes of symmetry of $ABCD$.




Let $Q$ be the perpendicular projection of $P$ onto $BC$, $M$ the midpoint of segment $AB$ and put $x = BQ$, $y = CQ$. The area of triangle $ABP$ is thus $6x$, and using the Pythagorean theorem in triangle $MBP$, we find $BP = \sqrt{x^2 + 6^2}$. The equality of area and perimeter of $\triangle ABP$ thus translates into the equation

$$6x = 12 + 2\sqrt{x^2 + 6^2}$$

with the only solution $x = 9/2$.

The value of $y$ can be found similarly: We have $BP = 15/2$ and $CP = \sqrt{y^2 + 6^2}$, so the condition on triangle $BCP$ implies

$$\frac{1}{2} \cdot 6 \cdot \left(y + \frac{9}{2}\right) = y + \frac{9}{2} + \frac{15}{2} + \sqrt{y^2 + 6^2}$$

with the only positive solution $y = 5/2$.

We conclude that $CP = 13/2$ and the perimeter of triangle $CDP$ equals 25.

**Problem 49.** The pair of integers $(0, 0)$ is written on a blackboard. In each step, we replace it in this way: If there is a pair $(a, b)$, we replace it with $(a+b+c, b+c)$, where either $c = 247$ or $c = -118$ (we may choose the number $c$ in each step). Find the smallest (non-zero) number of steps after which the pair $(0, b)$ for some $b$ appears on the blackboard.

*Result.* 145

*Solution.* Denote $c_i$ the number $c$ used in the $i$-th step. After $n$ steps, the number $a$ (i.e. the first coordinate of the pair) will be $a = nc_1 + (n-1)c_2 + \cdots + c_n$. Fix $n$ and let $s = n\varepsilon_1 + (n-1)\varepsilon_2 + \cdots + \varepsilon_n$, where $\varepsilon_i = 1$ if $c_i = 247$, and $\varepsilon_i = 0$ otherwise. Define $t$ in a similar fashion with $\varepsilon_i = 1$ if and only if $c_i = -118$. Clearly $a = 247s - 118t$, so the condition $a = 0$ implies $247s = 118t$, but as the numbers 247 and 118 are coprime, there is an integer $k$ such that $s = 118k$ and $t = 247k$. It follows that

$$365k = s + t = 1 + 2 + \cdots + n = \frac{n(n+1)}{2},$$

and since $365 = 5 \cdot 73$, we see that $n$ is at least $2 \cdot 73 - 1 = 145$.

It remains to show that there are numbers $c_i$ such that $247s = 118t$ with $n = 145$. Indeed, let $m$ be the smallest positive integer such that

$$1 + 2 + \cdots + m \geq \frac{247}{365} \cdot (1 + 2 + \cdots + n);$$

now put $c_i = -118$ for $i \in \{1, \ldots, m\} \setminus \{r\}$ and $c_i = 247$ otherwise, where

$$r = 1 + 2 + \cdots + m - \frac{247}{365} \cdot (1 + 2 + \cdots + n).$$

(In fact, $m = 120$ and $r = 97$.) This way we get precisely

$$247s = 118t = \frac{118 \cdot 247}{365} \cdot (1 + 2 + \cdots + n)$$

as desired.

**Problem 50.** A *zigzag* consists of two parallel rays of opposite directions with the initial points joined with a segment. What is the maximum number of regions the plane can be divided into using ten zigzags?

*Result.* 416

*Solution.* Every two zigzags can intersect in at most nine points, and for any number of zigzags, we may easily achieve the configuration when every two intersect in exactly nine points (and each point is the intersection of at most two lines). Consider placing the zigzags one by one: The $n$-th added zigzag is divided by the $9(n-1)$ intersection points with the $n-1$ already placed zigzags into $9(n-1)+1$ segments, each dividing an existing region into two. It follows the maximum number of regions definable using $n$ zigzags, $Z_n$, satisfies $Z_1 = 2$ and $Z_n = Z_{n-1} + 9n - 8$ for $n \geq 2$. The general result $Z_n = \frac{9}{2}n^2 - \frac{7}{2}n + 1$ then follows easily by induction and in particular, $Z_{10} = 416$.

**Problem 51.** Each face of a tetrahedron is a triangle with sides 1, $\sqrt{2}$, and $c$ and the circumradius of the tetrahedron is $5/6$. Find $c$.

*Result.* $\sqrt{23}/3$

*Solution.* We will prove a more general result: If each side of a tetrahedron is a triangle with sides $a$, $b$, $c$ and the circumradius of the tetrahedron is $\varrho$, then $a^2 + b^2 + c^2 = 8\varrho^2$. The result in our particular situation then follows directly by plugging in.

Inscribe the tetrahedron in a cuboid with edges of lengths $p$, $q$, $r$ so that the edges of the tetrahedron are the face diagonals of the cuboid. By the Pythagorean theorem,

$$p^2 + q^2 = a^2, \quad p^2 + r^2 = b^2, \quad \text{and} \quad q^2 + r^2 = c^2.$$

Furthermore, the circumsphere of the tetrahedron coincides with the one of the cuboid, the diameter of which is the space diagonal. Therefore

$$(2\varrho)^2 = p^2 + q^2 + r^2 = \frac{1}{2}(a^2 + b^2 + c^2),$$

which after rearranging gives the claimed equality.

**Problem 52.** For a big welcome party butler James has lined up 2016 cocktail glasses in a row, each containing delicious cherry cocktail. To finish things up, his task is to cover one of the glasses with a silver lid, to put a statue on top of the lid and to distribute an odd number of cherries into the uncovered glasses, at most one cherry per glass. How many possible arrangements of cherries and the lid are there if there have to be more cherries on the right-hand side of the lid than on its left-hand side?

*Result.* $2016 \cdot 2^{2013}$

*Solution.* First, consider all possible arrangements of the lid and at most one cherry in each uncovered glass without posing any further condition. There are 2016 possible locations for the lid and $2^{2015}$ possibilities to put at most one cherry in each of the remaining 2015 glasses, which gives a total of $2016 \cdot 2^{2015}$ arrangements. The binomial formula expansion

$$0 = (-1+1)^{2016} = \sum_{i=0}^{2016} \binom{2016}{i} (-1)^i = \sum_{i=0}^{1008} \binom{2016}{2i} - \sum_{i=1}^{1008} \binom{2016}{2i-1}$$

shows that the number of all arrangements having an even number of cherries is equal to the number of all arrangements having an odd number of cherries. Hence, the set $M$ of all arrangements of the lid and an odd number of cherries contains $\frac{1}{2} \cdot 2016 \cdot 2^{2015} = 2016 \cdot 2^{2014}$ elements. Now observe that every element $(n_1, n_2)$ of $M$ representing the arrangement with $n_1$ cherries on the left-hand side of the lid and $n_2$ cherries on its right-hand side has got exactly one corresponding element $(n_2, n_1)$ in $M$. These arrangements are different from each other, because the sum $n_1 + n_2$ being an odd number implies that one of the numbers $n_1$ or $n_2$ is even, the other one is odd, and one number is bigger than the other. Hence, exactly one of the two corresponding arrangements complies with the stated condition that there should be more cherries on the right-hand side of the lid than on its left-hand side. Therefore, the answer is $\frac{1}{2} \cdot 2016 \cdot 2^{2014} = 2016 \cdot 2^{2013}$.

**Problem 53.** We are given a wooden cube with its surface painted green. There are 33 different planes, each located between some two opposite faces of the cube and parallel to them, which dissect the cube into small cuboidal blocks. Given that the number of blocks with at least one green face equals the number of blocks with no green faces, determine the total number of blocks into which the cube is dissected.

*Result.* 1260 or 1344

*Solution.* It is easy to see that there have to be at least four planes in each of three possible directions (if in one of the directions there are less than five layers of blocks, then the number of at-least-one-green-side blocks is greater than the number of inside blocks). Denote the numbers of planes in different directions by $a+3$, $b+3$, $c+3$, where $a$, $b$, $c$ are positive integers. It follows that $(a+3)+(b+3)+(c+3)=33$, so $a+b+c=24$.

The problem condition can be rewritten as

$$(a+4)(b+4)(c+4)=2(a+2)(b+2)(c+2)$$

which yields $abc=240=2^4\cdot 3\cdot 5$ after simplification. Since $a+b+c$ is even, either (1) exactly one or (2) all three of numbers $a$, $b$, $c$ are even.

In the case (1), one of the numbers $a$, $b$, $c$ (w.l.o.g. $a$) must be divisible by 16 and since $a+b+c=24<2\cdot 16$, we have $a=16$. It follows that $b+c=8$ and $bc=15$, so $\{b,c\}=\{3,5\}$. We can now calculate the total number of blocks: $(a+4)(b+4)(c+4)=20\cdot 7\cdot 9=1260$.

In the case (2) we have w.l.o.g. $a=4x$, $b=2y$, $c=2z$ where $xyz=15$ and $2x+y+z=12$. The only possibility is $x=3$, $\{y,z\}=\{1,5\}$, which gives $(a+4)(b+4)(c+4)=16\cdot 6\cdot 14=1344$.

**Problem 54.** Given a positive integer $n$, let $p(n)$ be the product of non-zero digits of $n$. Find the largest prime divisor of the number $p(1)+\cdots+p(999)$.

*Result.* 103

*Solution.* Let $S=p(1)+\cdots+p(999)$. By expanding $A=(0+1+2+\cdots+9)(0+1+2+\cdots+9)(0+1+2+\cdots+9)$ one can see that $A$ would be the result if we multiplied by zero digits as well. Hence we have $S=(1+1+2+\cdots+9)(1+1+2+\cdots+9)(1+1+2+\cdots+9)-1$ because of the extra 1 which we do not want to count. So

$$S=46^3-1=(46-1)(46^2+46+1)=3^3\cdot 5\cdot 7\cdot 103$$

and the conclusion follows.

**Problem 55.** Let $(a_n)_{n=1}^{\infty}$ be a strictly increasing sequence of positive integers such that $9\mid a_{3k-2}$, $14\mid a_{3k-1}$, and $19\mid a_{3k}$ for all positive integers $k$. Find the smallest possible value of $a_{2016}$.

*Result.* 14478

*Solution.* We may assume that for all $n$ the value of $a_n$ is the smallest integer greater then $a_{n-1}$, which satisfies the divisibility condition. Observe that given $a_{3k}$ there are only two options for $a_{3k+3}$: Either $a_{3k+3}=a_{3k}+19$, or $a_{3k+3}=a_{3k}+38$. The latter occurs if and only if there are integers $c$, $d$, $5\le d\le c\le 9$, such that $9\mid a_{3k}+c$ and $14\mid a_{3k}+d$, since this implies $a_{3k+1}=a_{3k}+c$ and $a_{3k+2}=a_{3k}+14+d\ge a_{3k}+19$.

There are exactly $\binom{6}{2}=15$ pairs $(c,d)$ satisfying $5\le d\le c\le 9$. Since the numbers 9, 14, and 19 are pairwise coprime, the Chinese remainder theorem guarantees that for each such pair $(c,d)$ there is exactly one non-negative integer $a_{3k}$ less than $9\cdot 14\cdot 19$ such that $19\mid a_{3k}$, $9\mid a_{3k}+c$ and $14\mid a_{3k}+d$. Therefore there are exactly 15 terms $a_{3k}$ less than $9\cdot 14\cdot 19$ for which $a_{3k+3}=a_{3k}+38$. It is easy to see that the difference of no two of these terms is 19 and that $9\cdot 14\cdot 19-19$ is not such a number, which means that $a_{3\ell}=9\cdot 14\cdot 19$ for some $\ell$. From the fact that $a_{3k+3}=a_{3k}+38$ happens exactly 15 times we infer that $\ell=9\cdot 14-15=111$.

For the terms $a_n$ succeeding $a_{333}$, the remainders modulo 9, 14, and 19 is the same as for $a_{n-333}$, so we obtain the relation $a_{n+333}=a_n+9\cdot 14\cdot 19$. We may easily compute that $a_{18}=114$, and so

$$a_{2016}=a_{6\cdot 333+18}=6\cdot 9\cdot 14\cdot 19+114=14478.$$

**Problem 56.** Let $P$ be a point inside triangle $ABC$. Points $D$, $E$, $F$ lie on the segments $BC$, $CA$, $AB$, respectively, such that the lines $AD$, $BE$, $CF$ intersect in $P$. Given that $PA=6$, $PB=9$, $PD=6$, $PE=3$, and $CF=20$, find the area of triangle $ABC$.

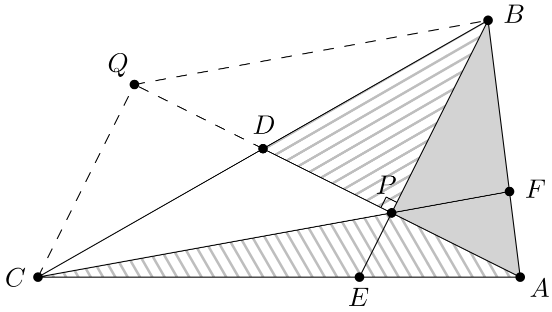
*Result.* 108

*Solution.* Denote by $[XYZ]$ the area of triangle $XYZ$. From $AP=DP$ we obtain $[ABP]=[BDP]$ and $[APC]=[DCP]$. Further, $3EP=BP$ implies that $3[APE]=[ABP]$ and

$$3[CEP]=[BCP]=[BDP]+[DCP]=3[APE]+[APE]+[CEP],$$

and so $[CEP]=2[APE]$. We conclude that $[ABP]=[BDP]=[APC]=[DCP]$; in particular, $BD=CD$.

Put $k = FP : CP$. Then from $AP = DP$ and $\angle APF = \angle CPD$ we have $[AFP] = k[DCP]$; similarly, $[FBP] = 3k[CEP]$. Combining with the known ratios above we get $k = 1/3$, therefore $FP = 5$, $CP = 15$. If we complete triangle $CPB$ to a parallelogram $CPBQ$, we may note that $BP^2 + PQ^2 = BQ^2$, and so $\angle DPB = 90°$.




We conclude that

$$[ABC] = 4[BDP] = 4 \cdot \frac{1}{2} \cdot 6 \cdot 9 = 108.$$

**Problem 57.** Find the last two digits before the decimal point of the number $(7 + \sqrt{44})^{2016}$.

*Result.* 05

*Solution.* Firstly observe that the number $7 - \sqrt{44}$ is strictly between 0 and 1, so the same holds for $(7 - \sqrt{44})^{2016}$. Moreover, the number $(7 + \sqrt{44})^{2016} + (7 - \sqrt{44})^{2016}$ is readily seen to be an integer using the binomial formula (the odd powers of $\sqrt{44}$ cancel out), so, in fact,

$$\lfloor (7 + \sqrt{44})^{2016} \rfloor = (7 + \sqrt{44})^{2016} + (7 - \sqrt{44})^{2016} - 1.$$

Exploiting the fact that $12^2 \equiv 44 \pmod{100}$, we obtain

$$(7 + \sqrt{44})^{2016} + (7 - \sqrt{44})^{2016} \equiv (7 + 12)^{2016} + (7 - 12)^{2016} \pmod{100},$$

so it suffices to find the last two digits of $19^{2016}$ and $5^{2016}$. The latter is just 25, as $5^3 \equiv 5^2 \pmod{100}$. To handle the former one, employ the binomial formula again to obtain

$$(20 - 1)^{2016} \equiv \binom{2016}{2015} \cdot 20^1 \cdot (-1)^{2015} + \binom{2016}{2016} (-1)^{2016} \equiv -19 \pmod{100}$$

(all the terms up to the last two ones are divisible by $20^2$). We conclude that the sought digits are $-19 + 25 - 1 = 05$.

*Alternative solution.* As above, we shall seek the last two digits of the number $(7 + \sqrt{44})^{2016} + (7 - \sqrt{44})^{2016}$. As the numbers $7 + \sqrt{44}$, $7 - \sqrt{44}$ are roots of the quadratic equation $x^2 - 14x + 5 = 0$, the sequences $(\alpha_n)_{n \geq 0}$, $(\beta_n)_{n \geq 0}$, defined via $\alpha_n = (7 + \sqrt{44})^n$ and $\beta_n = (7 - \sqrt{44})^n$, are subject to the recurrence relation $\alpha_{n+2} - 14\alpha_{n+1} + 5\alpha_n = 0$ and the same for $\beta_n$. Moreover, the same holds for their sum, $\gamma_n = (7 + \sqrt{44})^n + (7 - \sqrt{44})^n$. Our goal is to compute $\gamma_{2016} \bmod 100$.

Put $\tilde{\gamma}_n = \gamma_n \bmod 100$. The sequence $(\tilde{\gamma}_n)_{n \geq 0}$ is completely determined by the recurrence relation $\tilde{\gamma}_{n+2} = (14\tilde{\gamma}_{n+1} - 5\tilde{\gamma}_n) \bmod 100$ and the initial values $\tilde{\gamma}_0 = 2$, $\tilde{\gamma}_1 = 14$. Further, since $\tilde{\gamma}_n$ attains only finitely many values and every term depends only on the previous two, the sequence has to be periodic. By computing several of its values,

$$2, 14, 86, 34, 46, 74, 6, 14, 66, 54, 26, 94, 86, 34, \ldots,$$

we see that from $\tilde{\gamma}_2$ on, the sequence is periodic with period 10; thus $\tilde{\gamma}_{2016} = \tilde{\gamma}_6 = 6$. Since the sought number is one less, the last two digits are 05.